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A GENERALIZED APPROACH TO DIRECTION FINDING

Syracuse University

Hamza Ouibrahim and Donald Weiner

ROME AIR DEVELOPMENT CENTER
Air Force Systems Command
Griffiss Air Force Base, NY 13441-5700

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APPROVED: Vincent C. Vannicola

VINCENT C. VANNICOLA
Project Engineer

APPROVED: Frank J. Rehm

FRANK J. REHM
Technical Director
Directorate of Surveillance

FOR THE COMMANDER: John A. Ritz

JOHN A. RITZ
Directorate of Plans & Programs

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The problem of direction finding with a sensor array in a multiple source environment is addressed.

In this research a generalized formulation to the problem is developed. In the generalized formulation a matrix pencil is constructed. The rank reducing values are shown to contain the information needed to estimate the angles of arrival. It is also shown that several different nonsearch procedures are special cases of the generalized approach. The attractiveness of these techniques is their computational efficiency. The techniques differ depending upon the manner in which either the measured data is processed, or the angular information is extracted from the resulting equations. The pencil theorem establishes the relationship between the rank reducing values and the functional form generated by the operators applied to the measurements. This generalized approach allows for the inclusion of prefiltering before applying an operator to the measurements.

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Based on the generalized approach a new nonsearch direction of arrival estimation procedure is presented which is shown to be valid for the case of coherent sources.

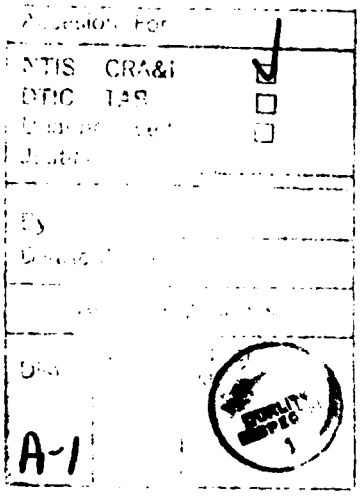

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER ONE

INTRODUCTION

1.1 Problem Statement

The problem of estimating the angular location of d sources using an array of m sensors is addressed. The sources are assumed to be in the far field with respect to the physical size of the array. The signals received by the sensors are generated by the sources themselves. For this reason, the direction finding system used in this work is classified as passive. The general system configuration is shown in Fig. 1.1.

It is assumed that the model which governs the signal received at the $i^{\underline{th}}$ sensor is of the form

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)a_i(\theta_k) + n_i(t) \; ; \; i=1,2,\ldots,m \qquad (1.1\text{-}1)$$

where $s_k(t)$ is the complex envelope of the $k^{\underline{th}}$ signal, $\underline{\theta}$ is the vector

$$\underline{\theta}^T = [\theta_1, \theta_2, \ldots, \theta_d]$$

where the $i^{\underline{th}}$ entry θ_i corresponds to the angular position of the $i^{\underline{th}}$ source, $a_i(\theta_k)$ is the response of the $i^{\underline{th}}$ sensor to the $k^{\underline{th}}$ source, and $n_i(t)$ is the additive noise. This noise is the sum of external and internal noise. External noise is assumed to be received uniformly from all directions while the internally generated thermal noise is assumed to be identically distributed in each of the m channels.

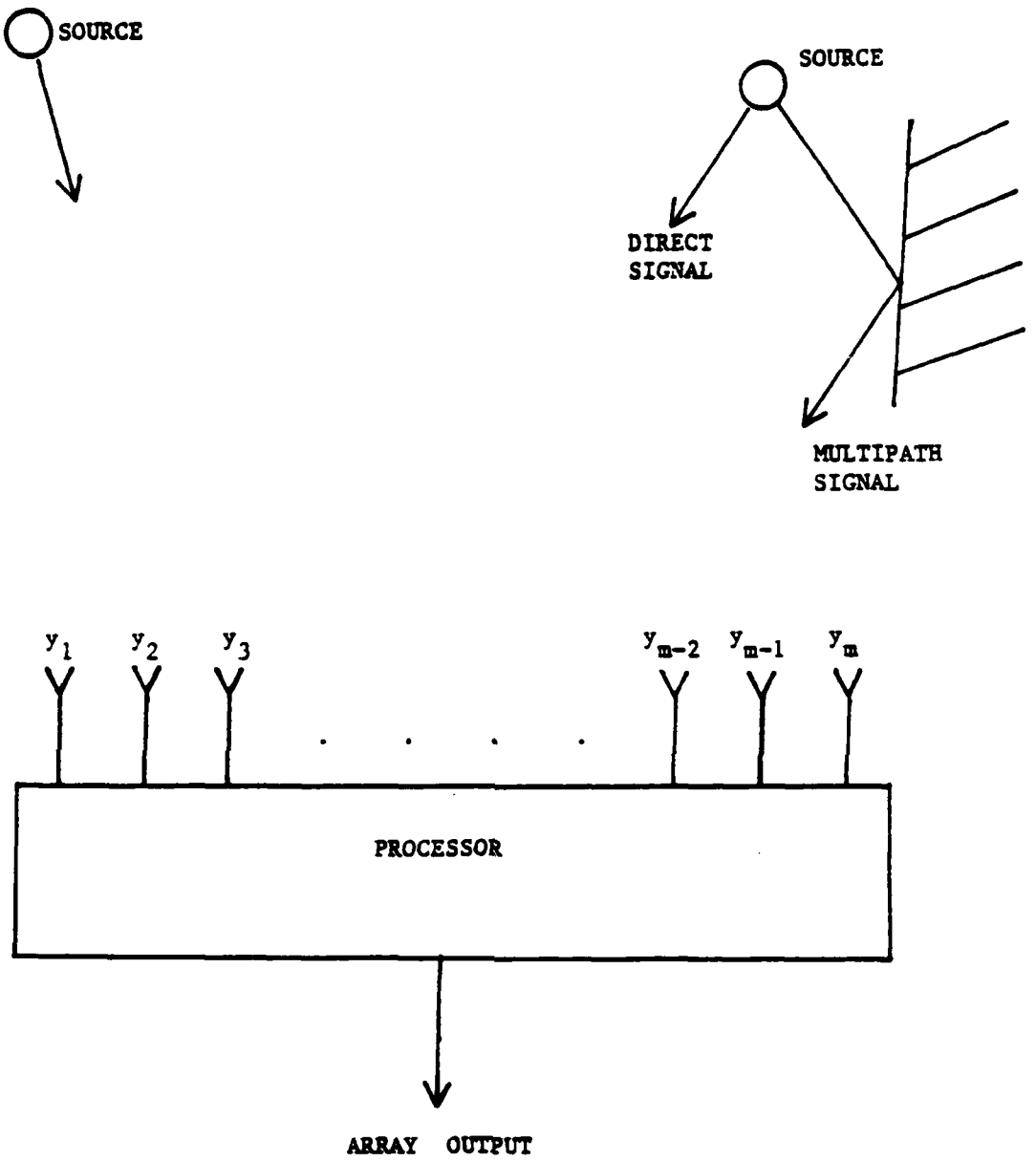

Fig. 1.1 Passive Direction Finding System

In this research the signals generated by the sources are assumed to be narrowband. A signal is classified as narrowband if its bandwidth is small compared to the inverse of the transit time of a wavefront across the array. Given N snapshots of noise corrupted data collected at the m sensors, the problem is to estimate the angular position of the d sources.

1.2 Previous Results

The problem of estimating the bearing or direction of arrival (DOA) of radiating sources has occupied many researchers in the last two decades. This problem has applications in areas such as sonar, radar, radio-astronomy, and seismology. Also, the performance of angle of arrival estimators plays an important role in determining the ultimate capability of any surveillance system. Originally, this problem was formulated in terms of the classical Bayesian and/or Neyman-Pearson decision theories [1], [2]. However, because this approach suffered from several deficiencies, a variety of new techniques have been proposed in recent years. In this section we review some of these methods.

Maximum Likelihood and Least-Squares Estimates

Two very popular approaches for constructing parameter estimates are the maximum likelihood and the least squares techniques. Application of the maximum likelihood approach to the direction finding problem is discussed by Böhme [3]. Let

$$\underline{Y} = [y_1(t,\underline{\theta}),\ y_2(t,\underline{\theta}),\ \ldots,\ y_m(t,\underline{\theta})]^T$$

denote the complex array data vector where $(\cdot)^T$ stands for the transpose operator. A maximum likelihood estimate (MLE) can be found provided both the probability density $f(\underline{Y}|\theta)$ of the data vector $\underline{Y}$ is known and the likelihood function

$$L(\underline{\theta}) = \log f(\underline{Y}|\underline{\theta})$$

can be maximized over the parameters

$$\underline{\theta} \triangleq [\theta_1, \theta_2, \ldots, \theta_d]^T.$$

An estimate $\hat{\underline{\theta}}$ maximizing $L(\underline{\theta})$ over $\underline{\theta}$ is defined to be an MLE of $\underline{\theta}$. Böhme points out that the MLE can be heavily biased. However, for independent and identically distributed random variables $y_1(t,\underline{\theta})$, $y_2(t,\underline{\theta}), \ldots, y_m(t,\underline{\theta})$, consistent MLE $\hat{\underline{\theta}}$ are asymptotically normally distributed and efficient. This means that the distribution of $\sqrt{m}\ (\hat{\underline{\theta}} - \underline{\theta})$ approaches the normal distribution with zero mean and covariance matrix given by the inverse of the Fisher information matrix

$$J = E_{\underline{\theta}} \left\{ \left(\frac{\partial \ln f(y_1|\underline{\theta})}{\partial \theta} \right) \left(\frac{\partial \ln f(y_1|\underline{\theta})}{\partial \underline{\theta}} \right)^H \right\}$$

where $(\cdot)^H$ stands for the Hermitian operator, $E_{\underline{\theta}}$ means expectation with respect to $\underline{\theta}$ and $\partial g/\partial \underline{\theta}$ is the column vector with elements $\frac{\partial g}{\partial \theta_i}$.

An estimate of $\underline{\theta}$ can also be obtained using the least squares approach. For the narrowband case, the data vector $\underline{Y}$ has the following form

$$\underline{Y} = A\underline{S} + \underline{N} \qquad (1.2\text{-}1)$$

where the (m x d) matrix A is the direction matrix whose columns $\{a(\theta_k), k = 1,2,\ldots,d\}$ are the signal direction vectors for the d wavefronts.

$\underline{S} \Delta [s_1(t), s_2(t),\ldots, s_d(t)]^T$ where $s_k(t)$ is the complex envelope of the $k^{\underline{th}}$ signal. The $(m \times 1)$ vector $\underline{N}$ represents the additive noise at the m sensors. In this approach one attempts to fit a signal model to the data vector $\underline{Y}$ in the least squares sense; that is, the error

$$E \underline{\Delta} \sum | \underline{Y} - A \underline{S} |^2$$

is minimized by choosing $\underline{S}$ and $\underline{\theta}$ [4]. If we set the derivatives of E with respect to the unknown parameters to zero in trying to minimize E, we realize that some of the necessary conditions are in the form of nonlinear equations. Attempting to solve this nonlinear set of equations is computationally very involved. A different approach to this minimization problem was suggested by several authors [5 - 8]. The idea is to choose values of $\underline{S}$ based on some a priori knowledge or based on some preprocessing and find the associated $\underline{\theta}$ that minimize E. Then the values of $\underline{S}$ can be altered to find new values for $\underline{\theta}$ that give a lower minimum for E. This process is repeated until a local minimum for E is found, and the corresponding $\underline{S}$ and $\underline{\theta}$ are the estimates of the unknown parameters. This is essentially a search procedure. H. Wang [9] comments that if no a priori information is available, the local minimum that is first reached may give an estimate which is far from the true values. Both approaches

discussed above prove to be computationally very involved tasks.

Beamforming

Beamforming is one of the oldest ideas in array processing for determining the bearing of a target. A simplified block diagram of a delay-and-sum beamformer is shown in Figure 1.2. The idea is to align the propagation delays of a signal presumed to be propagating in a direction $-\underline{k}$ so as to reinforce the signal [10]. For example, if the sensor delay τ_i; $i=1,2,\ldots,m$, is ideally adjusted to compensate for the signal delay $(\underline{Z}_i \cdot \underline{k})/c$, where $\underline{Z}_i$ is the spatial location of the $i^{\underline{th}}$ sensor and c is the speed of propagation, the signal power in the beam is m^2 times that measured at each sensor while the beam noise power is increased by only a factor of m (assuming the noise to be uncorrelated from sensor to sensor). Dudgeon [11] explains that the reason for studying the formation of beams from an array of sensors is to use the signals received by the sensors in a phased manner so as to preferentially detect signals coming from a particular direction. In addition, by averaging over many sensors, the signal to noise ratio (SNR) is increased to aid in the measurement of signal parameters [11].

The delay-and-sum beamformer consists of computing the energy in the beam for many directions of look by manipulating the delays. Maxima of this energy as a function of $\underline{k}$ will correspond to the location of the sources. Observe that this procedure requires computation of the energy in the beam at every direction of look. For this reason, this approach can be classified as a search procedure. The energy

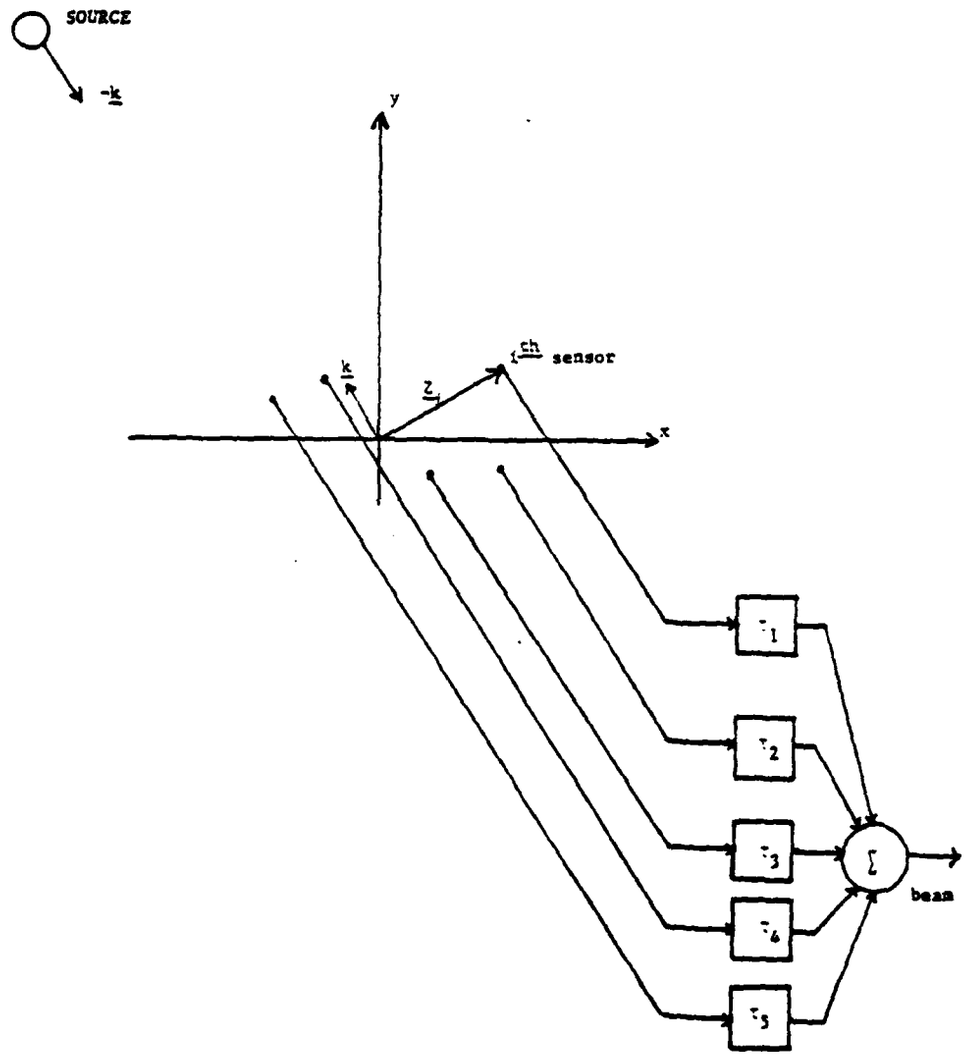

Fig. 1.2 Delay-and-Sum Beamformer where signal at i^{th} sensor is given by $y_i(t,\underline{\theta}) = s(t + \frac{\underline{z}_i \cdot \underline{k}}{c}) + n_i(t)$

in the beam when steered in direction $\underline{k}$ is given by

$$P(\underline{k}) = \underline{A}^H R \underline{A} \qquad (1.2\text{-}2)$$

where $R \triangleq E[\underline{Y}\underline{Y}^H]$ is the spatial correlation matrix of the sensor outputs and $\underline{A}$ is the direction of look vector with elements $A_i = \exp[j \frac{2\pi}{\lambda} \underline{k} \cdot \underline{Z}_i]$ where λ is the signal wavelength. Assuming the noise to be spatially white, the power in the beam when steered toward the source is

$$P(\underline{k}) = m^2\sigma_s^2 + m\sigma_n^2 \qquad (1.2\text{-}3)$$

where σ_s^2 and σ_n^2 denotes the power levels of the signal and noise. The value of $\underline{k}$ which yields the beam power given in (1.2-3) is also known as the Bartlett estimate of $\underline{k}$. For this value of $\underline{k}$, $P(\underline{k})$ is maximized.

The resolution of this approach is determined essentially by the beam pattern of the array of sensors. A typical beam pattern for a delay - and - sum beamformer is shown in Fig. 1.3. Using this classical method, increased bearing estimation accuracy can only be obtained by increasing the aperture of the array. This solution is of limited utility as it means increasing the physical size of the array. Don H. Johnson [10] observed that a source having a well defined bearing appears to be coming from a dominant but diffused direction as well as from false directions corresponding to sidelobes. The sidelobes are due to equal weighting assumed for each sensor output. This approach was shown to be incapable of resolving bearings spaced more

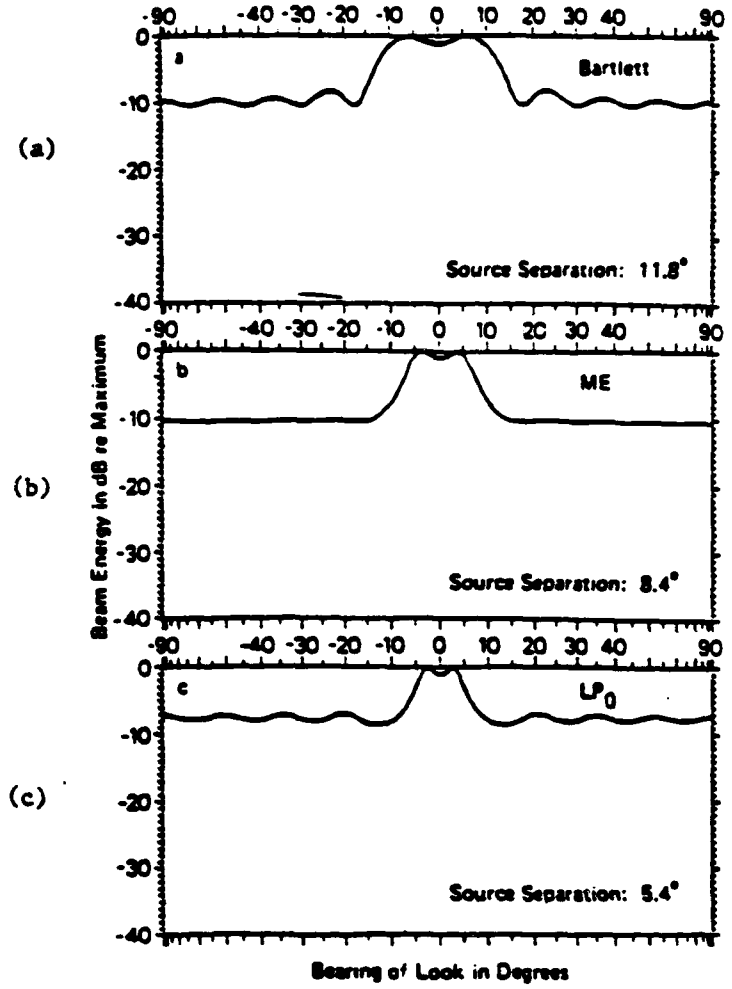

Fig. 1.3 Typical Beam Patterns.

(a) Delay-and-sum
(b) Maximum entropy
(c) Linear prediction

(From [12]).

closely than the Rayleigh limit regardless of SNR [12]. This minimum source separation below which sources cannot be resolved increases as the magnitude of the coherence between the sources increases.

Adaptive Beamforming

A way to alter the performance of the beamformer is to weight the sensor signals individually before summing them as shown in Figure 1.4. To achieve higher resolution modern spectral analysis algorithms are used which determine the weights by utilizing the data measured at the sensors. For this reason such systems are referred to as adaptive beamformers. The problem of determining the sensor weights so that the beam pattern has some desired characteristic is the same as designing a good data window for spectral estimation, or designing a prototype low-pass filter for use in a digital filter bank [11]. The sensor weights can be adjusted to maximize the SNR. This is analogous to designing a Wiener filter given the spectral estimates of the signal spectrum and noise spectrum [11]. J. P. Burg [13] notices that this filter will have a frequency response which passes those parts of the spectrum where the SNR is high and rejects those parts where the SNR is poor. This adaptive approach performs essentially the same as the delay-and-sum beamformer when the source angle separation is less than the beamwidth. Although this approach has better resolution capabilities than the conventional Fourier transform, it is still considered poor.

Spectral Estimation

The propagating waves are assumed to generate a homogeneous random field (i.e., a field that is space stationary). In this case a

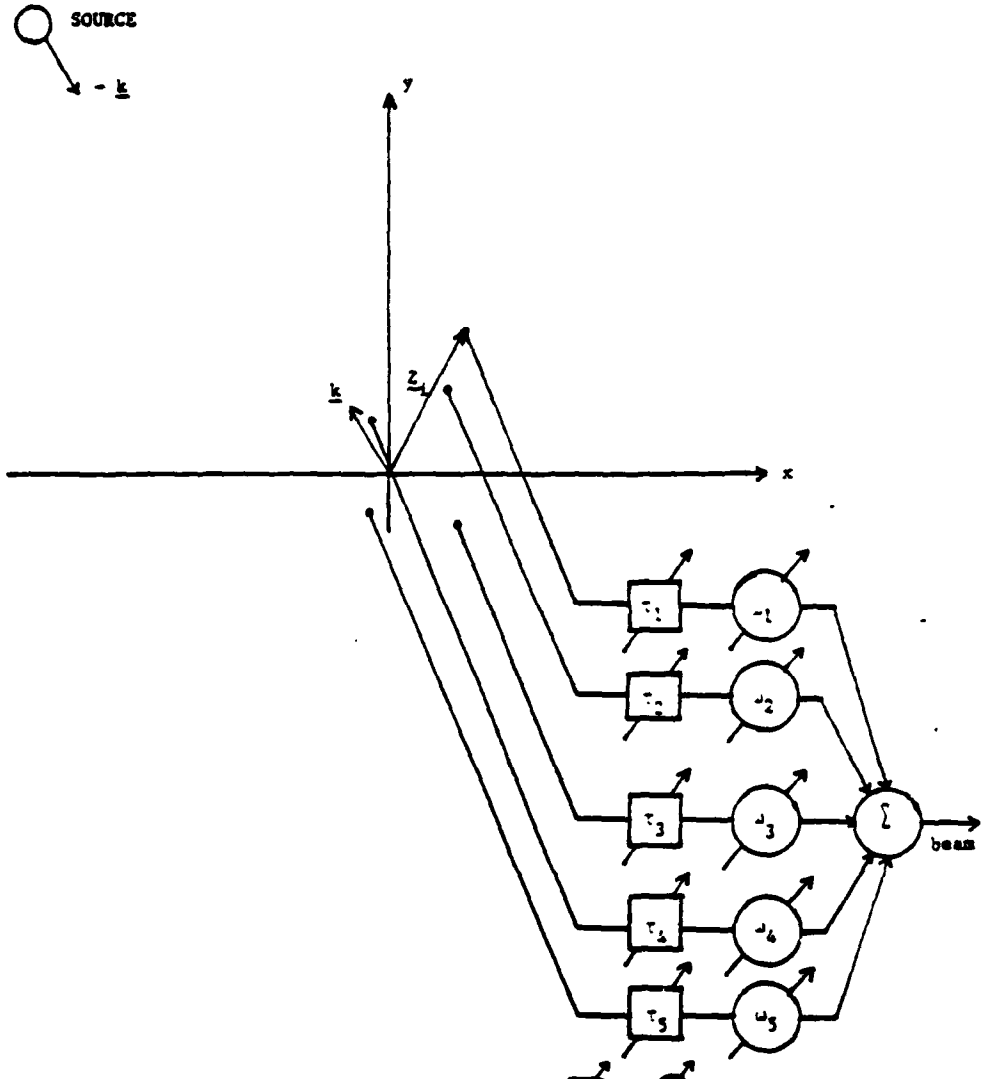

Fig. 1.4 Adaptive beamforming. The symbols and indicate variable phase shifters and attenuators, respectively.

spectral representation for the field exists, similar to that for stationary random processes, which consists of a superposition of traveling waves [14]. This, in effect, explains the equivalence between the problem of determining the bearing of a radiating source with an array of sensors and the problem of estimating the spectrum of a signal. D. Johnson gives an excellent review of this equivalence [10].

Linear prediction techniques can also be applied to the direction of arrival (DOA) estimation. A tutorial review of linear prediction is given by J. Makhoul [15]. Ulrych and Clayton [16] as well as Tufts and Kumaresan [17] have applied the forward-backward linear prediction method to the narrowband linear array case. Among the most well known high resolution spectral estimation methods are Capon's maximum likelihood, Burg's maximum entropy and the linear predictive method. We shall give a brief discussion of these three techniques as applied to DOA estimation.

Capon's maximum likelihood method does not make use of the standard MLE. Instead, a constrained optimization problem is solved. The direction of look $\underline{A}$ is found which yields the minimum beam energy $P(\underline{k})$ given in (1.2-2) subject the the constraint $\underline{A}^H\underline{E} = 1$, where $\underline{E}$ represents an ideal plane wave corresponding to the direction of look. This constraint fixes the processing gain to 1 for each direction of look. Minimizing the beam energy tends to reduce the contributions to this energy from sources and noise not propagating in the direction of

look [10]. For this reason S. R. Degraaf and D. H. Johnson called this approach minimum energy adaptive beamforming. Using the Lagrangian approach, the solution of the above stated optimization problem is given by

$$\underline{A} = \frac{R^{-1}\underline{E}}{\underline{E}^H R^{-1}\underline{E}} \qquad (1.2\text{-}4)$$

where R is the spatial correlation matrix of the data vector $\underline{Y}$ which was defined previously. This approach is classified as a search procedure since the vector $\underline{A}$ has to be computed for each direction of look. When the noise is assumed to be white and the beam is steered toward the source, the beam power estimate is

$$P(\underline{k}) = \sigma_s^2 + \frac{\sigma_n^2}{m}$$

where σ_s^2 and σ_n^2 denote the power levels of the signal and noise.

Burg [18] suggested using the principal of maximum entropy to estimate the DOA's. For a given direction of look, the entropy

$$H = \int \ln P(\underline{k})\, d\underline{k} \qquad (1.2\text{-}5)$$

of the power spectrum $P(\underline{k})$ is obtained. Burg has shown that maximizing the entropy is equivalent to choosing the "most likely" spectrum. For the case of an equally spaced linear array, the maximum entropy solution was shown to be equivalent to the linear prediction solution.

S. W. Lang and J. H. Mcllelan [19], among others, applied linear prediction (LP) to array processing problems. The idea is to estimate the output of the ℓ^{th} sensor as a linear combination of the other sensor outputs

$$y_\ell(t,\underline{\theta}) = - \sum_{i \neq \ell} u_i y_i(t,\underline{\theta}). \qquad (1.2\text{-}6)$$

The linear predictive coefficients are obtained by again solving a constrained optimization problem, the constraint being $u_\ell = 1$. Once these coefficients are known, the power spectrum $P(\underline{k})$ is obtained. The power spectrum $P(\underline{k})$ is computed for different directions of look and the source bearings correspond to the maximas. Observe that this is also a search procedure. An open question in the LP approach is the choice of ℓ, i.e. the choice of the sample to be predicted by the other samples.

The three methods described above are capable of resolving closely spaced source bearings if the SNR is large enough. Using incoherent sources and a linear array containing 10 sensors, it was shown that the linear prediction processing algorithm is uniformly the most capable of resolving closely-spaced signals [12]. Thus, these methods have better resolution capabilities than the beamformers presented earlier. However, one major drawback of these high resolution procedures is they are computationally very complex due to their search nature. Finally it should be noted that spectral estimation methods are not applicable to the DOA estimation

problem when the source signals are correlated because the wavefield generated by correlated signals is not homogeneous.

Signal-Subspace Processing

In recent years, there has been a great deal of interest in spectral estimation procedures based on an eigenvalue-eigenvector decomposition of the spatial correlation matrix. In the mathematical literature, this is known as principal component analysis. R. Kumaresan [20] points out that principal component analysis involves linearly transforming the data along the first few principal eigenvectors of the covariance matrix of the data [21]. The original work in this area is due to H. Hotelling [22]. C. R. Rao [23] showed that several problems in multivariate analysis have solutions based on calculating the first few eigenvectors of a correlation matrix. In array processing, interest stems from the fact that there are relations between the eigenvectors of the spatial correlation matrix and the directions of arrival. Pisarenko [24] was the first to realize the important properties of the eigenvectors of the correlation matrix. This idea was then vigorously developed by Cantoni and Godora [25], Owsley [26], Liggett [27], Reddi [28], Schmidt [29] and Bienvenu [30].

All these signal-subspace approaches assume the background noise to be independent from sensor to sensor and the spatial coherence matrix to be the identity matrix. It should be noted that when the noise is nonwhite but has known covariance, the problem can still be handled through prewhitening. A Paulraj and T. Kailath [31] proposed a

solution to the unknown noise covariance problem when the noise field is invariant under different array positions (i.e., the noise covariance matrix is unchanged). Situations where this assumption is valid are not uncommon in sonar applications.

We now present the building block of a standard eigenstructure algorithm. Using (1.2-1), the spatial covariance matrix can be written as

$$R = E[\underline{Y}\,\underline{Y}^H] = ASA^H + \sigma^2 I \qquad (1.2\text{-}7)$$

where S is the covariance matrix of the signals $\underline{s}(t)$, i.e.

$$S = E[\underline{s}(t)\,\underline{s}^H(t)],$$

σ^2 is an unknown constant representing the noise power at each sensor, and $E(\cdot)$ denotes expectation. Let $\{\lambda_1 > \lambda_2 > \ldots > \lambda_m\}$ and $\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_m\}$ be the eigenvalues and the corresponding eigenvectors of R. One can show that if S is nonsingular (i.e. the source signals are not coherent), and if the number of sources d is less than the number of sensors m, then

1) The minimum eigenvalue of R is σ^2 with multiplicity (m-d), i.e.

$$\lambda_{d+1} = \lambda_{d+2} = \ldots = \lambda_m = \sigma^2, \qquad (1.2\text{-}8)$$

2) The eigenvectors corresponding to the minimal eigenvalues are orthogonal to the columns of the matrix A or the signal direction vectors. In particular,

$$\{\underline{e}_{d+1}, \underline{e}_{d+2}, \ldots, \underline{e}_m\} \perp \{\underline{a}_k, k=1,2,\ldots,d\} \qquad (1.2\text{-}9)$$

where the symbol $\perp$ denotes orthogonality.

The key steps of this algorithm are as follows. First determine the number of sources d using (1.2-8). Next find the noise subspace as the space of the eigenvectors corresponding to the smallest eigenvalue of R. Finally, the desired direction vectors (hence the DOA's) are found by searching the array manifold to locate those vectors that are orthogonal to this noise subspace [29]. Like earlier approaches, these signal-subspace techniques are also search procedures.

The above algorithm assumes that we have perfectly known eigenvectors and eigenvalues. In practice, however, R has to be estimated from the array data vector $\underline{Y}$. Due to errors in estimating R from finite samples and because of finite precision arithmetic, the eigenvalues and eigenvectors will be perturbed from their true values. There is, therefore, zero probability that the smallest eigenvalue of R will have multiplicity (m-d), i.e. the small eigenvalues will all be different with probability one. A more sophisticated approach, based on statistical considerations, is needed to determine the underlying multiplicity [32]. The problem of estimating the number of sources present is an on going one. Many solutions have been proposed but none is fool proof. Liggett [27] was the first researcher to use eigenvalues of the estimated R to determine the number of signal sources. He fitted a multivariate signal model to the observed data. Bartlett [33] and Lawley [34] developed a procedure based on a nested sequence

of hypothesis tests. For each test the likelihood ratio statistic is computed and compared to a threshold. The hypothesis accepted is the first one for which the threshold is crossed. The problem with this method is the subjective judgment required for deciding on the threshold levels [32]. One of the most celebrated approaches to this problem is the Aikake information theoretic criteria (AIC). The advantage of this approach is that no objective judgment is required in the decision process [35]. The minimum description length (MDL) is another approach that does not require objective judgment. Recently, a new approach based on the AIC and MDL has been proposed by M. Wax and T. Kailath [32]. They view the problem as a model selection problem and then apply the AIC or MDL for model selection.

To close this subsection we present one of the most promising subspace approaches, called Multiple Signal Classification (MUSIC), which was proposed by Schmidt [36]. MUSIC can be shown to provide asymptotically unbiased estimates of

1) number of incident wavefronts present
2) directions of arrival
3) strengths and cross correlations among the incident waveforms.
4) noise/interference strength
5) polarizations

Given the m x m spatial covariance matrix R of the data vector $\underline{Y}$, Schmidt showed that the eigenvectors associated with the minimum

eigenvalues of R are orthogonal to the space spanned by the columns of A, i.e. the signal direction vectors. Thus, Schmidt proposed searching in the $\underline{a}(\theta)$ continuum for the values of $\underline{a}(\theta)$ which will maximize the expression

$$P(\theta) = \frac{1}{\underline{a}^H(\theta) E_N E_N^H \underline{a}(\theta)} \qquad (1.2\text{-}10)$$

where E_N is defined to be the m x (m-d) matrix whose columns are the (m-d) noise eigenvectors. These new eigenvectors are the eigenvectors corresponding to the (m-d) smallest eigenvalues. Once the signal direction vectors are known, the DOA's can then be obtained. Note that MUSIC necessitates searching. Schmidt generalized this approach to include the added complexity of signal polarization.

Schmidt compared MUSIC with the conventional beamformer, Capon's maximum likelihood and Burg's maximum entropy approaches (See Fig. 1.5). MUSIC was found to have superior bias, error variance and resolution performance. However, like all the earlier search methods, this super-resolution approach is very inefficient computationally.

Nonsearch Procedures

Currently, nonsearch procedures are being developed [37], [38], [39]. These algorithms seem to have important advantages over search procedures. Among these advantages are [37]:

1) They are computationally less complex because a search procedure is not needed.
2) These algorithms do not require knowledge of element characteristics (i.e. directional pattern, gain/phase response).

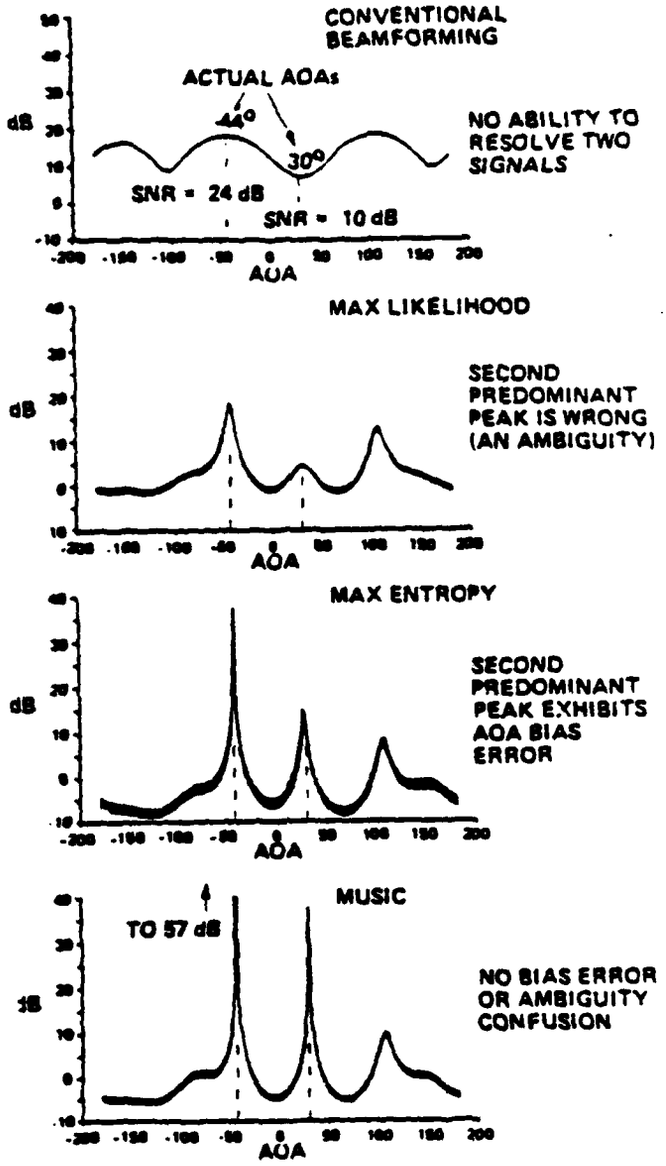

Fig. 1.5 Example of Azimuth Only Direction Finding Performance. (From [29]).

3) They do not require a calibration of the array. This completely eliminates need for storage of the array manifold which can be very large for multidimensional problems.

This dissertation concerns itself with these newly developed nonsearch procedures. Although the techniques are developed for narrowband signals, wideband signals can also be handled by decomposing them into narrowband signal sets using comb filters. A discussion of DOA estimation of wideband sources is given by H. Wang and M. Kaveh [40].

1.3 Research

Nonsearch methods for solving the problem of direction finding with a sensor array in a multiple source environment are the main topic of this research

Chapter 2 is devoted to formulating a generalized approach for these nonsearch procedures. The generalized approach consists of applying an operator to the received signals in order to form a matrix pencil $M-\lambda N$. The rank reducing values of λ are shown to contain the information needed to estimate the angles of arrival. The pencil theorem proposed in this chapter establishes the relationship between the rank reducing values of λ and the functional form $f(\phi_i)$ generated by the operators applied to the measurements. When the matrices M and N are square, the rank reducing values of λ are the generalized eigenvalues of the pencil $M-\lambda N$. Two methods are proposed

to compute these values of λ when M and N are nonsquare.

Several different nonsearch procedures are special cases of the generalized approach. These methods differ primarily depending upon the operator used. In chapter 3 two different techniques are presented and formulated in terms of the generalized framework. The first, ESPRIT, makes use of a phase delay operator while the second makes use of a summation operator.

In chapter 4 the moving window operator is analyzed in detail. This operator is also formulated within the generalized framework and is shown to hold for coherent sources. At first the moving window is presented for the case of deterministic signals. In section 4.1-2 the moving window is shown to be applicable to the zero mean random signal case. In section 4.2 the moving window operator is applied to a rectangular planar array. As before, the rank reducing numbers of the matrix pencil $M-\lambda N$ contain the information needed to locate the sources. Sections 4.3 and 4.4 present Prony's and Pisarenko's algorithms. Their relationship to the moving window is developed. In section 4.5 computer simulation results are presented. The estimation accuracy of the Prony, Pisarenko, and moving window methods are evaluated in terms of bias and variance based on several Monte Carlo runs. The moving window is also compared to ESPRIT.

Finally, a summary and suggestions for future work are given in chapter 5.

CHAPTER 2

THE GENERALIZED APPROACH

As discussed in the introduction, several high resolution spatial processing algorithms such as the maximum likelihood method, maximum entropy method, linear predictive method, and MUSIC have been proposed for estimating the directions of arrival of incoming signals. These techniques are essentially search procedures and, because of that, are computationally very complex. Nonsearch procedures have recently been proposed to alleviate this computational burden.

In this chapter a generalized formuation is proposed for these newly developed techniques. The pencil theorem which forms the foundation of this work is developed in section 2.1. This theorem states that the rank reducing values of a certain matrix pencil contain the information needed to estimate the angular location of the d sources. Section 2.1 discusses the procedures one can use to obtain these rank reducing numbers.

2.1 The Pencil Theorem

The basic problem under consideration is that of estimating the angular location of d sources given measurements from an array of m sensors. The measurements are modeled by a linear combination of d exponentials whose exponents $j\phi_i, i = 1,2,\ldots, d$ contain the information about the angular location of the sources. The generalized approach is based on the concept of a matrix pencil which is defined as follows [41]. Let M and N be two k x ℓ matrices. Also, denote

the space of complex numbers by $\mathbb{C}$. The set of all matrices of the form $M - \lambda N$ with $\lambda \in \mathbb{C}$ is said to be a pencil.

The matrices M and N are required to have the decompositions:

$$M = EF \text{ and } N = EDF \tag{2.1-1}$$

where E is a $k \times d$ matrix ($k \geq d$), F is a $d \times \ell$ matrix ($\ell \geq d$), D is a $d \times d$ diagonal matrix, and E, D, and F are all of rank d. In addition, the $ii^{\underline{th}}$ entry of the diagonal matrix D is required to be of the form

$$d_{ii} = f(\phi_i) = f[g(\theta_i)]; \; i = 1,2,\ldots d \tag{2.1-2}$$

where θ_i is the angular location of the $i^{\underline{th}}$ source. The matrices M and N and the function $f(\cdot)$ are determined by the operator $T\{\cdot\}$ applied to the signals received at the m sensors. In general, different approaches may employ different operators $T\{\cdot\}$.

The following theorem establishes the relationship between the values of λ for which the rank of the pencil $M - \lambda N$ decreases by 1 and the functions $f(\cdot)$ which contain the directions of arrival (DOA) information.

Pencil Theorem:

Let $M - \lambda N$ be a matrix pencil where M and N have the decompositions

$$M = EF \text{ and } N = EDF$$

as cited above. Then, the values of λ which decrease the rank of the pencil $M - \lambda N$ by 1 are given by

$$\lambda_i = \{f(\phi_i)\}^{-1} = \{f[g(\theta_i)]\}^{-1} \quad ; i=1,\ldots,d.$$

Proof:

Given the matrix pencil $M - \lambda N$, it follows that

$$M - \lambda N = EF - \lambda EDF = E(I - \lambda D)F \tag{2.1-3}$$

where the $ii^{\underline{th}}$ entry of the diagonal matrix $(I-\lambda D)$ is given by

$$[(I - \lambda D)]_{ii} = 1 - \lambda f(\phi_i).$$

If $\lambda = \{f(\phi_i)\}^{-1}$, then the $i^{\underline{th}}$ column of $(I - \lambda D)$ becomes zero. The rank of $(I - \lambda D)$, denoted by $r[I - \lambda D]$, is thus reduced by 1 to $(d-1)$. In general

$$r[M-\lambda N] = r[E(I - \lambda D)F] = \min \{r[E], r[I- \lambda D], r[F]\}.$$

However, by assumption

$$r[E] = r[F] = d.$$

Therefore, the rank of the pencil $M - \lambda N$ is decreased by 1 when

$$\lambda_i = \{f(\phi_i)\}^{-1} \quad ; i = 1,2,\ldots d. \tag{2.1-4}$$

2.2 Evaluation of the Rank Reducing Numbers

The procedure used to compute the λ_i's is now discussed. It is shown to depend on whether M and N are square or non-square matrices. Denote the determinant of a matrix M by det(M), and the space of complex numbers by $\mathbb{C}$. In case M and N are SQUARE matrices, the

generalized eigenvalues (G.E.) of the pencil $M - \lambda N$ are defined to be all the elements $\lambda_i \in \mathbb{C}$ of the set $\lambda(M,N)$ defined by

$$\lambda(M,N) = \{\lambda \in \mathbb{C} \ni \det (M - \lambda_i N) = 0\}.$$

When the G.E.'s are distinct, the rank of $M - \lambda N$ is reduced by 1 whenever λ is set equal to one of the G.E.'s. Therefore, these G.E.'s are the rank reducing numbers of the square matrix pencil $M - \lambda N$.

If M and N happen to be NONSQUARE matrices, then either one of two approaches can be used to determine the rank-reducing numbers λ_i:

1. The problem can always be reduced to a generalized eigenvalue problem by premultiplying the rectangular matrix pencil $M - \lambda N$ by M^H. Note that $(\cdot)^H$ denotes the Hermitian operator. We obtain

$$M^H(M - \lambda N) = M^H M - \lambda M^H N. \tag{2.2-1}$$

Observe that $M^H M - \lambda M^H N$ is a square matrix pencil. It follows from (2.1-1) that

$$\begin{aligned} M^H M - \lambda M^H N &= (EF)^H (EF) - \lambda (EF)^H (EDF) \\ &= F^H E^H EF - \lambda F^H E^H EDF \\ &= F^H E^H E \, (I - \lambda D) F. \end{aligned} \tag{2.2-2}$$

Note in the above equation that we still have the decomposition required by the pencil theorem. The matrices $F^H E^H E$ and F are both of rank d. Because $(I - \lambda D)$ arises in both (2.1-3) and (2.2-2), the

generalized eigenvalues of the matrix pencil $M^H M - \lambda M^H N$ are identical to those obtained in (2.1-4) for the matrix pencil $M-\lambda N$ when M and N are square.

2. A Grammian approach can also be used to determine the λ_i's. Denote the $i^{\underline{th}}$ column of $M - \lambda N$ by $(M - \lambda N)_i$. Also, define the inner product of two vectors $\underline{X}$ and $\underline{Y}$ by

$$<\underline{X}, \underline{Y}> = \underline{Y}^H \underline{X}. \qquad (2.2\text{-}3)$$

This approach consists of checking the dependence/independence of the columns of the matrix pencil $M - \lambda N$. To this end a Gram matrix G whose $ij^{\underline{th}}$ entry is given by

$$G_{ij} = <(M - \lambda N)_i, \ (M - \lambda N)_j>$$

is formed. Computing the determinant of G results in a polynomial $P(\lambda)$ whose non-zero zeros are the rank-reducing numbers of $M - \lambda N$.

Having shown how the rank reducing numbers of a matrix pencil can be determined, we now show that different nonsearch procedures for DOA estimation can be formulated within this generalized framework.

CHAPTER 3

VARIOUS OPERATORS FOR USE IN THE GENERALIZED APPROACH

The rank reducing numbers of the matrix pencil $M - \lambda N$ will be functions of the angular locations of the d sources provided the $i^{\underline{th}}$ diagonal element of the matrix D has the functional form given by (2.1-2). This functional form, however, is not dictated by the location of the sources but by the operator used to process the data received at the sensors. Different methods formulated within the framework of the generalized approach use different oprators.

In this chapter two direction of arrival estimation techniques are presented. The first, ESPRIT, which makes use of a delay operator, is discussed in section 3.1. The second approach, presented in section 3.2, is a generalization of a system identification scheme originally proposed by Jain [42]. This scheme uses a summation operator.

3.1 Phase Delay Operator (ESPRIT)

ESPRIT [37] is a subspace approach to direction of arrival estimation which employs a planar array of arbitrary geometry composed of m matched sensor doublets whose elements are translationally separated by a known constant displacement vector Δ (See Fig. 3.1). In terms of the generalized formulation, it will be shown that ESPRIT uses a phase delay operator which results in the function

$$f(\phi_i) = e^{-j\phi_i}. \qquad (3.1\text{-}1)$$

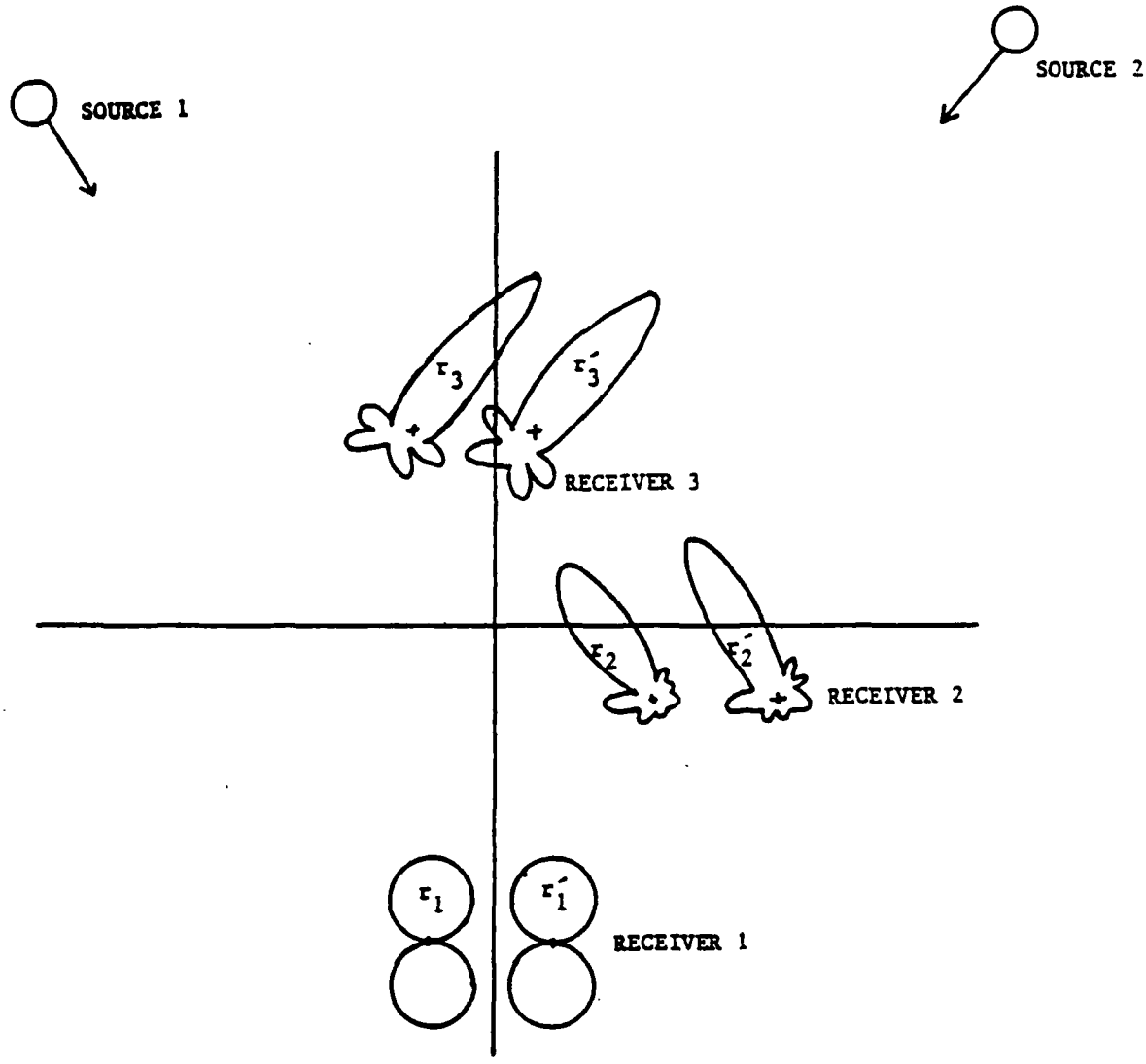

Fig. 3.1 Multiple Source DOA Estimation Using ESPRIT.

(From [37])

The array of sensors is conveniently described as being comprised of two subarrays X and Y, identical in every aspect although physically displaced from each other by a known displacement vector Δ. Assume there are d narrowband stationary sources located at azimuthal angles θ_i, i=1,2,...d, which are impinging on the array as planar wavefronts and emitting signals whose complex envelopes are denoted by $s_k(t)$, k= 1, 2,...,d. To be able to solve for the DOA's d has to be less than m. The signal received by the $i^{\underline{th}}$ sensor is the superposition of d impinging wavefronts plus zero-mean additive noise. Define $a_i(\theta_k)$ to be the relative response of the $i^{\underline{th}}$ sensor at either subarray to the $k^{\underline{th}}$ source, ω_0 to be the center frequency of each of the spatial sources, c to be the speed of propagation of the plane waves, and $n_{x_i}(t)$ and $n_{y_i}(t)$ to be the additive noise at the elements in the $i^{\underline{th}}$ doublet. Then the output of the $i^{\underline{th}}$ sensor can be expressed as,

$$x_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\, a_i(\theta_k) + n_{x_i}(t)$$

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\, a_i(\theta_k) e^{j\phi_k} + n_{y_i}(t)$$

where $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_d)$

and $$\phi_k = g(\theta_k) = \frac{\omega_0}{C} \Delta \sin \theta_k. \qquad (3.1\text{-}2)$$

Let

$$\underline{X}^T(t) = [x_1(t,\underline{\theta}), x_2(t,\underline{\theta}), \ldots, x_m(t,\underline{\theta})]$$

$$\underline{Y}^T(t) = [y_1(t,\underline{\theta}), y_2(t,\underline{\theta}), \ldots, y_m(t,\underline{\theta})]$$

be simultaneously sampled data vectors of both subarrays. Similarly, define

$$\underline{s}^T(t) = [s_1(t), s_2(t),\ldots, s_d(t)]$$

$$\underline{n}_x^T(t) = [n_{x_1}(t), n_{x_2}(t),\ldots, n_{x_m}(t)]$$

$$\underline{n}_y^T(t) = [n_{y_1}(t), n_{y_2}(t),\ldots,n_{y_m}(t)].$$

Define A to be the m x d direction matrix whose columns are the signal direction vectors for the d wavefronts defined by

$$\underline{a}_k^T(\theta_k) = [a_1(\theta_k),a_2(\theta_k),\ldots, a_m(\theta_k)];\ k=1,\ldots d.$$

Also, define Φ to be the diagonal matrix of the phase delays between the doublet sensors given by

$$\Phi = \text{diag}\,[e^{j\phi_1}, e^{j\phi_2},\ldots, e^{j\phi_d}]. \qquad (3.1\text{-}3)$$

It follows that

$$\underline{X}(t) = A\,\underline{S}(t) + \underline{n}_x(t)$$

$$\underline{Y}(t) = A\,\Phi\,\underline{S}(t) + \underline{n}_y(t). \qquad (3.1\text{-}4)$$

As a first step in this algorithm the minimum description length (MDL) criterion [32] is used to estimate d, the number of sources. Using both this estimate and the assumption that the noise is white and uncorrelated from sensor to sensor, the maximum likelihood estimate of the noise power is obtained. The matrix

pencil C-λB is then formed where C is the autocorrelation of the data received by subarray X and B is the cross-correlation matrix between the data received at subarrays X and Y. Note that C and B are square correlation matrices. Assuming that $\underline{S}(t)$, $\underline{n}_x(t)$ and $\underline{n}_y(t)$ are statistically independent, then

$$\begin{aligned} C &= E[\underline{X}(t)\ \underline{X}^H(t)] \\ &= E[(A\ \underline{S}(t) + \underline{n}_x(t))(A\ \underline{S}(t) + \underline{n}_x(t))^H] \\ &= A\ S\ A^H + R_{\underline{n}_x \underline{n}_x}(0) \end{aligned} \tag{3.1-5}$$

where

$$S = E[\underline{S}(t)\ \underline{S}^H(t)] \tag{3.1-6}$$

is the d x d correlation matrix of the signals s(t),

$$R_{\underline{n}_x\ \underline{n}_x}(0) = E[\underline{n}_x(t)\ \underline{n}_x^H(t)]$$

is the d x d correlation matrix of the noise and $E(\cdot)$ denotes the expected value.

Let

$$E = AS \text{ and } F = A^H.$$

Using these expressions C can be rewritten as

$$C = EF + R_{\underline{n}_x \underline{n}_x}(0) \tag{3.1-7}$$

Define

$$D = \Phi^H = \text{diag}\ [e^{-j\phi_1}, e^{-j\phi_2}, \ldots, e^{-j\phi_d}]. \tag{3.1-8}$$

The matrix B can be expressed as

$$B = E[\underline{X}(t)\ \underline{Y}^H(t)]$$

$$= E[(AS(t) + \underline{n}_x(t))(A\Phi\underline{S}(t) + \underline{n}_y(t))^H]$$

$$= AS\Phi^H A^H + R_{\underline{n}_x\underline{n}_y}(0)$$

$$= EDF + R_{\underline{n}_x\underline{n}_y}(0) \qquad (3.1\text{-}9)$$

where

$$R_{\underline{n}_x\underline{n}_y}(0) = E[\underline{n}_x(t)\ \underline{n}_y^H(t)].$$

Since the noise components are assumed to be white and uncorrelated from sensor to sensor, it follows that

$$R_{\underline{n}_x\underline{n}_x}(0) = \sigma^2 I \text{ and } R_{\underline{n}_x\underline{n}_y}(0) = 0.$$

Therefore,

$$C - \lambda B = E(I - \lambda D)F + \sigma^2 I. \qquad (3.1\text{-}10)$$

Let

$$M = (C - \sigma^2 I) = E[\underline{X}(t)\underline{X}^H(t)] - \sigma^2 I = EF$$

and

$$N = B = E[\underline{X}(t)\underline{Y}^H(t)] = EDF. \qquad (3.1\text{-}11)$$

Subtraction of σ^2 I from both sides of (3.1-10) then yields

$$(C - \sigma^2 I) - \lambda B = E(I-\lambda D)\ F = M - \lambda N. \qquad (3.1\text{-}12)$$

Observe that the matrix pencil formed in this manner has the decomposition required by the generalized formulation.

Recall that the m x d matrix E is the product of the direction matrix A and the signal covariance matrix S. The matrix E is required to be of full rank. This, in turn, requires that the matrices A and S both be of rank d. The m x d matrix A is of rank d as long as the signal angles of arrival are distinct. However, the rank of S will be less than d when the signals are coherent. This explains why ESPRIT fails in the case of coherent signals. The matrix F is of full rank, as required, since it is the matrix A^H. The diagonal matrix D given in (3.1-8) is also of rank d.

As mentioned in (3.1-1) the phase delay operator of ESPRIT results in

$$f(\phi_i) = e^{-j\phi_i}.$$

From the pencil theorem it follows that the G.E.'s of M-λN are related to $f(\phi_i)$ by

$$\lambda_i = \{f(\phi_i)\}^{-1} = e^{j\phi_i};\ i = 1,2,\ldots,d.$$

Once the G.E.'s are known, the DOA's can be obtained from (3.1-2) using the relation

$$\theta_i = \arcsin\left\{-j\,\frac{c}{\omega_0 \Delta}\,\ell n\lambda_i\right\}.$$

Note that the argument of arcsin is real since $\ell n\ \lambda_i$ is purely imaginary.

It can be concluded that this nonsearch procedure is one application of the generalized approach. The operator used by ESPRIT is the phase

delay operator which results in the functional relationship given in (3.1-1). ESPRIT has all the advantages discussed in the introduction, i.e., speed, storage, and indifference to calibration.

3.2 Summation Operator

In this section we generalize a system identification scheme first proposed by V. K. Jain [42]. In this generalization a linear array composed of m identical omnidirectional sensors with uniform spacing Δ is used to estimate the direction of arrival of far field point sources. Assume there are $d \leq m/2$ narrowband deterministic sources with spectra centered at ω_0 and located far enough from the array such that their wavefronts impinging on the array are planar. θ_k is the direction of arrival of the k^{th} source and $n_i(t)$ is the additive noise at the i^{th} sensor. Additive noise is present at all m sensors. The signal received at the i^{th} sensor is expressed as

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\, e^{j\frac{\omega_0}{c}\Delta(i-1)\sin\theta_k} + n_i(t);\ i=1,2,\ldots,m$$

As before, $s_k(t)$ denotes the complex envelope of the signal emitted by the k^{th} source.

For a reason which will become apparent later the i^{th} measurement $y_i(t,\underline{\theta})$ is weighted by a known decaying exponential $e^{-b(i-1)}$. The weighted signal is given by

$$\begin{aligned}\omega_i(t,\underline{\theta}) &= e^{-b(i-1)} y_i(t,\underline{\theta}) \\ &= \sum_{k=1}^{d} s_k(t)\, e^{-b(i-1)}\, e^{j\frac{\omega_0}{c}\Delta(i-1)\sin\theta_k} + e^{-b(i-1)} n_i(t).\end{aligned}$$

Let

$$\phi_k = g(\theta_k) = \exp\left[-b + j\frac{\omega_0}{c}\Delta \sin\theta_k\right] \tag{3.2-1}$$

Hence, $\omega_i(t,\underline{\theta})$ can be expressed as

$$\omega_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)[g(\theta_k)]^{(i-1)} + e^{-b(i-1)}\, n_i(t).$$

Assuming the signals to be deterministic and the noise to be zero mean, $x_i(t,\underline{\theta})$ is defined as

$$x_i(t,\underline{\theta}) = E[\omega_i(t,\underline{\theta})] = \sum_{k=1}^{d} s_k(t)\,[g(\theta_k)]^{(i-1)}. \tag{3.2-2}$$

Define $x_1(i,t,\underline{\theta})$ as follows

$$x_1(i,t,\underline{\theta}) = x_i(t,\underline{\theta}) \;;\; i=1,\ldots,m. \tag{3.2-3}$$

A vector $\underline{X}_1$, whose $i^{\underline{th}}$ entry is given by $x_1(i,t,\underline{\theta})$, is then formed as indicated below (See Fig.3.2),

$$\underline{X}_1^T = \{x_1(1,t,\underline{\theta}),\, x_1(2,t,\underline{\theta}),\ldots,\, x_1(m,t,\underline{\theta})\}\ . \tag{3.2-4}$$

As in previous techniques the number of signals, d, is estimated. Having an estimate d, a set of (d+1) vectors $\underline{X}_p$, p = 1,2,...d+1 is created by operating successively (p-1) times on $\underline{X}_1$ with the summation operator, $\sum_{\ell=j}^{m}(\cdot)$. For example, the elements of the vector $\underline{X}_2$ or $x_2(j,t,\underline{\theta})$, j = 1,2,...m are obtained as shown below:

$$x_2(j,t,\underline{\theta}) = \sum_{\ell=j}^{m} x_1(\ell,\, t,\, \underline{\theta}). \tag{3.2-5}$$

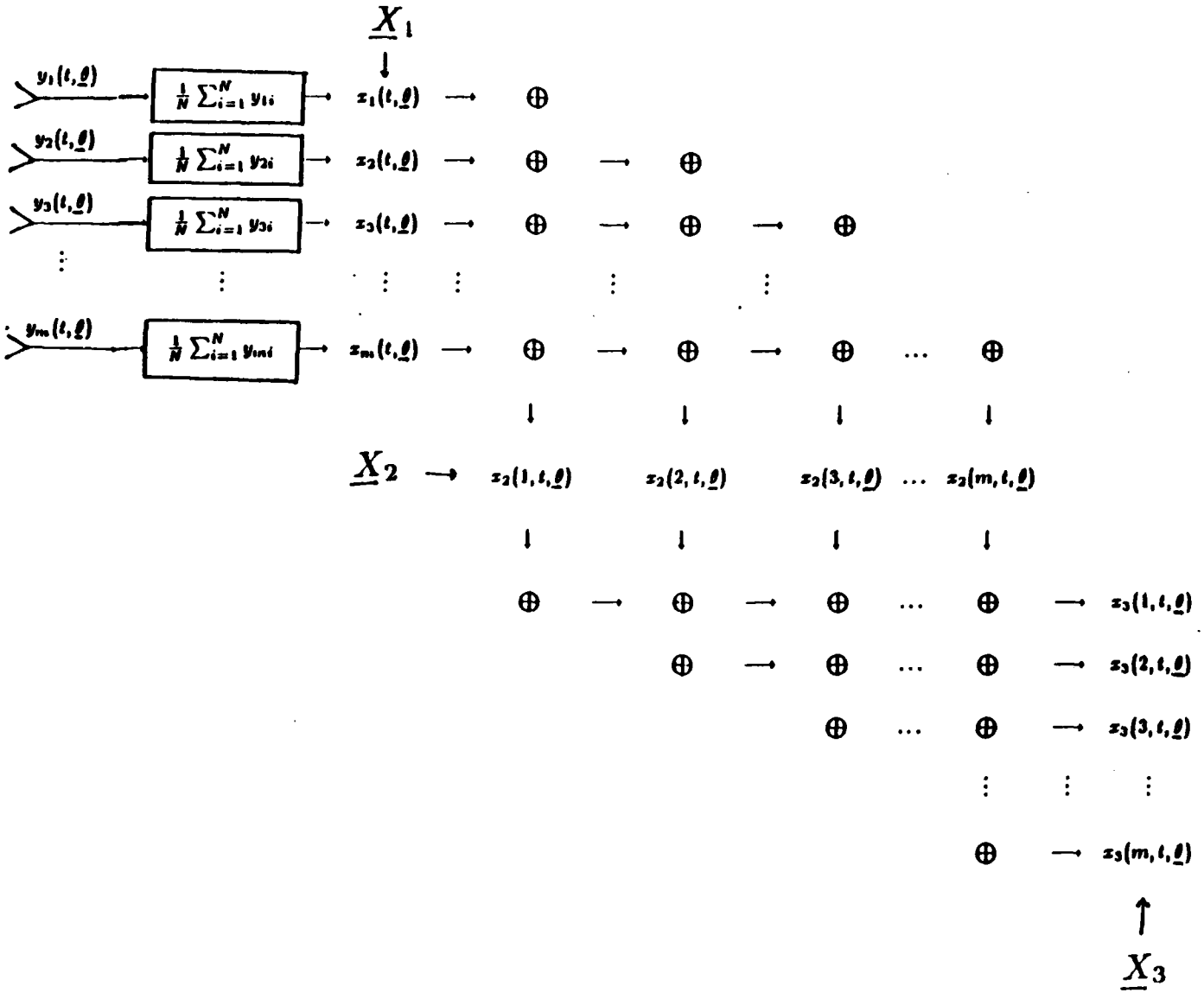

Fig. 3.2 Summation Operator

Using (3.2-2) and (3.2-3), (3.2-5) can be rewritten as

$$x_2(j,t,\underline{\theta}) = \sum_{\ell=j}^{m} \sum_{k=1}^{d} s_k(t)\,[g(\theta_k)]^{(\ell-1)} = \sum_{k=1}^{d} \frac{s_k(t)}{g(\theta_k)} \sum_{\ell=j}^{m} [g(\theta_k)]^{\ell}$$

$$= \sum_{k=1}^{d} \frac{s_k(t)}{g(\theta_k)} \{ \sum_{\ell=0}^{m} [g(\theta_k)]^{\ell} - \sum_{\ell=0}^{j-1} [g(\theta_k)]^{\ell} \}$$

$$= \sum_{k=1}^{d} \frac{s_k(t)}{g(\theta_k)} \{ \frac{1 - [g(\theta_k)]^{(m+1)}}{1 - g(\theta_k)} - \frac{1 - [g(\theta_k)]^{j}}{1 - g(\theta_k)} \}$$

$$= \sum_{k=1}^{d} \frac{s_k(t)}{1-g(\theta_k)} [g(\theta_k)]^{(j-1)} - \sum_{k=1}^{d} \frac{s_k(t)}{1-g(\theta_k)} [g(\theta_k)]^{m}.$$

Similarly, the elements of the vector $\underline{X}_3$ or $x_3(j,t,\underline{\theta})$ $j = 1,2,\ldots,m$ are obtained by using the summation operator on $\underline{X}_2$.

$$x_3(j,t,\underline{\theta}) = \sum_{\ell=j}^{m} x_2(\ell,t,\underline{\theta})$$

$$= \sum_{\ell=j}^{m} \sum_{k=1}^{d} \frac{s_k(t)}{1-g(\theta_k)} [g(\theta_k)]^{(\ell-1)} - \sum_{\ell=j}^{m} \sum_{k=1}^{d} [g(\theta_k)]^{m}$$

$$= \sum_{k=1}^{d} \frac{s_k(t)}{[1-g(\theta_k)]^2} [g(\theta_k)]^{(j-1)} - \sum_{k=1}^{d} \frac{s_k(t)}{[1-g(\theta_k)]^2} [g(\theta_k)]^{m}$$

$$- \sum_{k=1}^{d} \frac{s_k(t)}{1-g(\theta_k)} (m-j+1)\,[g(\theta_k)]^{m}.$$

This procedure continues until the vector $\underline{X}_{d+1}$ is obtained. Define

$$C_1(j) = \sum_{\ell=j}^{m} (m-\ell+1)$$

$$C_2(j) = \sum_{=j}^{m} C_1(\ell) \qquad (3.2\text{-}6)$$

$$\vdots$$

$$C_{d-2}(j) = \sum_{\ell=j}^{m} C_{d-2}(\ell).$$

The elements of $\underline{X}_{d+1}$, or $x_{d+1}(j,t,\underline{\theta})$, are given by

$$x_{d+1}(j,t,\underline{\theta}) = \sum_{\ell=j}^{m} x_d(\ell,t,\underline{\theta}). \qquad (3.2\text{-}7)$$

Using (3.2-6) expression (3.2-7) becomes

$$x_{d+1}(j,t,\underline{\theta}) = \sum_{k=1}^{d} \frac{s_k(t)}{[1-g(\theta_k)]^d} [g(\theta_k)]^{(j-1)}$$

$$- \sum_{k=1}^{d} \frac{s_k(t)}{[1-g(\theta_k)]^d} [g(\theta_k)]^m - \sum_{k=1}^{d} \frac{s_k(t)}{[1-g(\theta_k)^{d-1}} (m-j+1) [g(\theta_k)]^m$$

$$- \sum_{k=1}^{d} \frac{s_k(t)}{[1-g(\theta_k)]^{d-2}} C_1(j) [g(\theta_k)]^m$$

$$- \sum_{k=1}^{d} \frac{s_k(t)}{[1-g(\theta_k)]^{d-3}} C_2(j) [g(\theta_k)]^m$$

$$\vdots$$

$$- \sum_{k=1}^{d} \frac{s_k(t)}{[1-g(\theta_k)]} C_{d-2}(j) [g(\theta_k)]^m.$$

In general, the $j^{\underline{th}}$ element of the $q^{\underline{th}}$ vector has the following form

$$x_q(j,t,\underline{\theta}) = \sum_{k=1}^{d} \frac{s_k(t)}{[1-g(\theta_k)]^{q-1}} [g(\theta_k)]^{(j-1)} - \sum_{k=1}^{d} h(j,t,m,\theta_k) [g(\theta_k)]^m \tag{3.2-8}$$

where $h(j,t,m,\theta_k)$ is a function containing all the terms multiplying $[g(\theta_k)]^m$. Assume that the weights are chosen such that $|g(\theta_k)|^m = \varepsilon << 1$. Note that

$$|g(\theta_k)|^m = \exp(-bm) = \varepsilon.$$

Therefore,

$$b = - \ln \frac{\varepsilon}{m} .$$

Assuming b is chosen such that the terms involving $[g(\theta_k)]^m$ are negligible with respect to the first term in the right side of (3.2-8), $x_q(j,t,\underline{\theta})$ $q = 1,2,\ldots,d+1$ and $j = 1,2,\ldots,m$ can be approximated as follows:

$$x_q(j,t,\underline{\theta}) \approx \sum_{k=1}^{d} \frac{s_k(t)}{[1-g(\theta_k)]^{q-1}} [g(\theta_k)]^{j-1}. \tag{3.2-9}$$

As a parenthetical note, it is pointed out that the above result is obtained with equality for a hypothetical array containing an infinite number of sensors. Then the upper limit in the summation operator is ∞ and the terms including $[g(\theta_k)]^m$ do not appear. However, even in the case $m = \infty$, this technique requires the measurements to be weighted so that terms involving the summation operator converge.

Using the vectors $\underline{X}_i$; i=1,2,..., d+1, the matrix pencil $M-\lambda N$ is formed where the rectangular matrices M and N are defined to be

$$M = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \underline{X}_1 & \underline{X}_2 & \cdots & \underline{X}_d \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \qquad N = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \underline{X}_2 & \underline{X}_3 & \cdots & \underline{X}_{d+1} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}. \quad (3.2\text{-}10)$$

Let

$$E = \begin{bmatrix} 1 & 1 & 1 & \cdots\cdots & 1 \\ g(\theta_1) & g(\theta_2) & \cdots & & g(\theta_d) \\ g^2(\theta_1) & g^2(\theta_2) & \cdots & & g^2(\theta_d) \\ \vdots & \vdots & & & \vdots \\ g^{m-1}(\theta_1) & g^{m-1}(\theta_2) & \cdots & & g^{m-1}(\theta_d) \end{bmatrix} \quad (3.2\text{-}11)$$

and

$$F = \begin{bmatrix} s_1(t) & s_1(t)/1-g(\theta_1) & \cdots & s_1(t)/[1-g(\theta_1)]^{d-1} \\ s_2(t) & s_2(t)/1-g(\theta_2) & \cdots & s_2(t)/[1-g(\theta_2)]^{d-1} \\ \vdots & \vdots & & \vdots \\ s_d(t) & s_d(t)/1-g(\theta_d) & \cdots & s_d(t)/[1-g(\theta_d)]^{d-1} \end{bmatrix}. \quad (3.2\text{-}12)$$

Using the above matrices, the matrix M can be decomposed as

$$M = EF. \quad (3.2\text{-}13)$$

Similarly, define the diagonal matrix

$$D = \text{diag}\left[\frac{1}{1-g(\theta_1)} , \frac{1}{1-g(\theta_2)} , \ldots , \frac{1}{1-g(\theta_d)} \right].$$

It follows that N can be decomposed as

$$N = EDF. \tag{3.2-14}$$

Using the decompositions (3.2-13) and (3.2-14) it is seen that the matrix pencil generated by this approach takes the required form given in (2.1-3). Provided the directions of arrival are distinct, E and F are of full rank. Consequently, the summation operator approach generates a matrix pencil that satisfies the requirements of the pencil theorem.

In the summation operator approach the $ii^{\underline{th}}$ entry d_{ii} of the matrix D is given by

$$d_{ii} = \frac{1}{1-g(\theta_i)} .$$

In terms of the generalized formulation

$$f(\theta_i) = \frac{1}{1-\phi_i} = \frac{1}{1-g(\theta_i)} \; ; \; i = 1,2,\ldots,d.$$

The values of λ which decrease the rank of the pencil $M - \lambda N$ by 1 are given by (2.1-4) or

$$\lambda_i = \{f(\phi_i)\}^{-1} = 1-g(\theta_i) \; ; \; i=1,2,\ldots,d. \tag{3.2-15}$$

Because the matrices M and N are not square in this approach, the λ_i's can be obtained by computing the G.E.'s of the square matrix pencil

given by (2.2-1). Alternatively, the Gram matrix approach can be used to obtain these rank reducing numbers.

Jain's system identification scheme has been generalized to handle the direction finding problem. It is shown that this method, like ESPRIT, is an application of the generalized approach. This scheme makes use of the summation operator. Being a nonsearch procedure, it has all the advantages ESPRIT presents.

There is still another operator, the moving window operator, to be discussed in greater detail in chapter four.

CHAPTER 4

THE MOVING WINDOW OPERATOR

Two operators, the phase delay operator and the summation operator, were presented in chapter three. Both methods were formulated in terms of the generalized approach. Being nonsearch procedures, these methods possess the computational and storage advantages discussed earlier. However, the ESPRIT method that makes use of a phase delay operator fails for coherent sources. The summation operator, on the other hand, required a weighting factor to approximately eliminate terms that would have prevented the matrix decomposition required.

In this chapter another operator is presented. This operator, the moving window, is formulated within the generalized framework and is shown to hold for coherent sources. This approach provides asymptotically unbiased estimates of the angles of arrival. Section 4.1 introduces the moving rectangular window for the case of deterministic signals. In subsection 4.1-1 the case of a singular signal covariance matrix is analyzed. The rectangular moving window is then applied to the zero-mean random signal case in subsection 4.1-2. In section 4.2 the moving rectangular window is generalized to a rectangular planar array. Sections 4.3 and 4.4 present two well known methods, Prony's and Pisarenko's methods [42-46]. Their relationship to the moving rectangular window is developed. A comparative performance based on computer simulation of the three methods is given in section 4.5.

4.1 The Moving Rectangular Window

In this section a nonsearch procedure is presented for azimuth-only

DOA estimation of far field point sources. Consider a linear array composed of m identical omnidirectional sensors with uniform sensor spacing D. Assume there are $d \leq m/2$ narrowband deterministic sources located at azimuthal angles θ_i, $i = 1,2,\ldots,d$, which are impinging on the array as planar wavefronts and emitting signals whose complex envelopes are denoted by $s_k(t)$, $k=1,2,\ldots,d$. The signal received at the $i^{\underline{th}}$ sensor is the superposition of d impinging wavefronts plus zero-mean additive noise. Define $a_i(\theta_k)$ to be the relative response of the $i^{\underline{th}}$ sensor to the k^{th} source, ω_0 to be the center frequency of each of the spatial sources, c to be the speed of propagation of the plane waves, and $n_i(t)$ to be the additive noise of the $i^{\underline{th}}$ sensor. Then the signal received at the $i^{\underline{th}}$ sensor can be expressed as

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\ a_i(\theta_k) + n_i(t);\ i=1,2,\ldots,m.$$

Let

$$\phi_k = \frac{\omega_0}{c}\ D \sin \theta_k\ ;\ k=1,2,\ldots,d. \qquad (4.1\text{-}1)$$

When using a linear array of omnidirectional sensors, the relative response (sensor 1 being the reference sensor) of the $i^{\underline{th}}$ sensor to the $k^{\underline{th}}$ source is given by

$$a_i(\theta_k) = e^{j\phi_k\ (i-1)}. \qquad (4.1\text{-}2)$$

$y_i(t,\underline{\theta})$ can then be rewritten as

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\ \exp[\{j\phi_k\}\ (i-1)] + n_i(t)$$

$$= \sum_{k=1}^{d} s_k(t)\ \exp[(j\frac{\omega_0}{c}\ D \sin \theta_k)\ (i-1)] + n_i(t). \qquad (4.1\text{-}3)$$

The expected value of $y_i(t,\underline{\theta})$ yields $x_i(t,\underline{\theta})$ given by

$$x_i(t,\underline{\theta}) = E[y_i(t,\underline{\theta})]$$

$$= \sum_{k=1}^{d} s_k(t)\, a_i(\theta_k);\ i=1,2,\ldots,m. \qquad (4.1\text{-}4)$$

Define the rectangular window to be

$$R_N(n) = \begin{cases} 1 & 1 \le n \le N \\ 0 & \text{otherwise.} \end{cases} \qquad (4.1\text{-}5)$$

Given the number of sources d and the m averaged data points $x_i(t,\underline{\theta})$, a set of (d+1) vectors $\underline{X}_n$ is created where the components of $\underline{X}_n$ are the (m-d) values of the sequence

$$x_i(t,\underline{\theta})R_{m-d}\,(i-n+1);\ n = 1,2,\ldots,d+1$$

$$n \le i \le m + n - d - 1. \qquad (4.1\text{-}6)$$

In particular, (see Fig. 4.1)

$$\begin{aligned} \underline{X}_1^T &= \{x_1(t,\underline{\theta}),\ x_2(t,\underline{\theta}),\ldots,\ x_{m-d}(t,\underline{\theta})\} \\ \underline{X}_2^T &= \{x_2(t,\underline{\theta}),\ x_3(t,\underline{\theta}),\ldots,\ x_{m-d+1}(t,\underline{\theta})\} \\ &\vdots \\ \underline{X}_{d+1}^T &= \{x_{d+1}(t,\underline{\theta}),\ x_{d+2}(t,\underline{\theta}),\ldots,x_m(t,\underline{\theta})\}. \end{aligned} \qquad (4.1\text{-}7)$$

The matrix pencil $M-\lambda N$ is then formed where

$$M = \left[\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \underline{X}_1 & \underline{X}_2 \cdots & \underline{X}_d \\ \downarrow & \downarrow & \downarrow \end{array}\right] \qquad N = \left[\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \underline{X}_2 & \underline{X}_3 \cdots & \underline{X}_{d+1} \\ \downarrow & \downarrow & \downarrow \end{array}\right] \qquad (4.1\text{-}8)$$

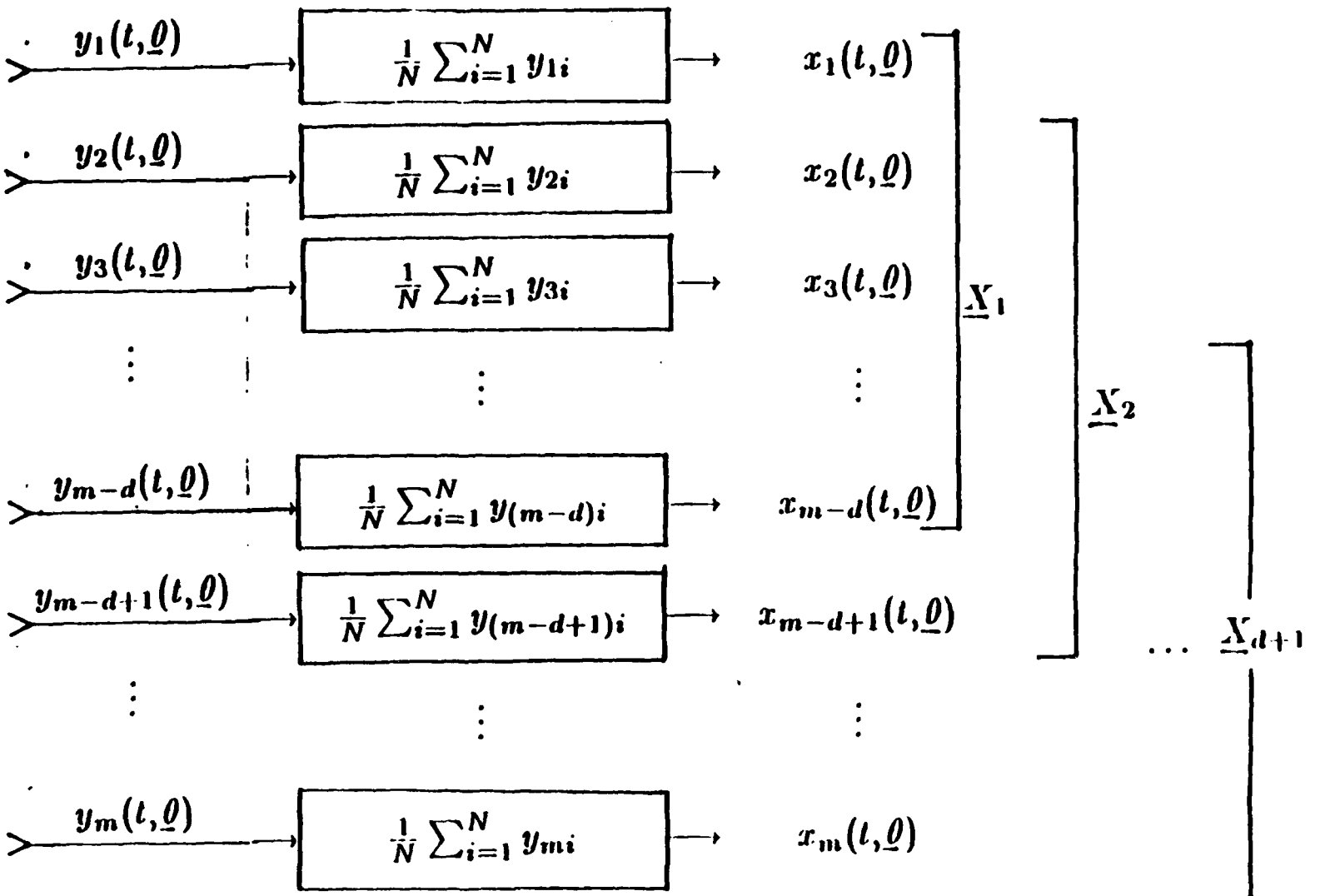

Fig. 4.1 Moving Window Operator

Define the matrices

$$A_n = \begin{bmatrix} a_n(\theta_1) & a_n(\theta_2) & \cdots & a_n(\theta_d) \\ a_{n+1}(\theta_1) & a_{n+1}(\theta_2) & \cdots & a_{n+1}(\theta_d) \\ \vdots & \vdots & & \vdots \\ a_{n+(m-d-1)}(\theta_1) & a_{n+(m-d-1)}(\theta_2) & \cdots & a_{n+(m-d-1)}(\theta_d) \end{bmatrix} \tag{4.1-9}$$

$$E = A_n\Big|_{n=1} = A_1 \tag{4.1-10}$$

and

$$\begin{aligned} \Phi &= \text{diag}\,[d_{11}, d_{22}, \ldots, d_{dd}] \\ &= \text{diag}\,[e^{j\phi_1}, e^{j\phi_2}, \ldots, e^{j\phi_d}]. \end{aligned} \tag{4.1-11}$$

Also, define the vector

$$\underline{S}^T = [s_1, s_2, \ldots, s_d].$$

It can be shown that the vector $\underline{X}_n$ can be expressed as

$$\underline{X}_n = A_n \underline{S};\ n=1,2,\ldots,d+1. \tag{4.1-12}$$

Observe from (4.1-2) that

$$a_n(\theta_k) = \{a_2(\theta_k)\}^{(n-1)} = \{e^{j\phi_k(n-1)}\}.$$

It follows from the above observation that

$$A_n = A_1\Phi^{(n-1)} = E\Phi^{(n-1)}.$$

Consequently,

$$\underline{X}_n = A\underline{S} = E\Phi^{(n-1)}\underline{S}. \tag{4.1-13}$$

Define

$$F = \begin{bmatrix} s_1 & s_1 e^{j\phi_1} & \cdots & s_1 e^{j(d-1)\phi_1} \\ s_2 & s_2 e^{j\phi_2} & \cdots & s_2 e^{j(d-1)\phi_2} \\ \vdots & \vdots & & \vdots \\ s_d & s_d e^{j\phi_d} & \cdots & s_d e^{j(d-1)\phi_d} \end{bmatrix}. \tag{4.1-14}$$

The matrix M can then be expressed as

$$M = EF \tag{4.1-15}$$

where E and F are given by (4.1-10) and (4.1-14), respectively. In a similar manner N can be expressed as

$$N = E\Phi F. \tag{4.1-16}$$

Using (4.1-15) and (4.1-16), the matrix pencil $M-\lambda N$ can be decomposed as

$$M - \lambda N = E(I - \lambda\Phi)F. \tag{4.1-17}$$

This decomposition satisfies the requirements of the pencil theorem. As long as the directions of arrival of the signals are distinct and the separation is less than half a wavelength, the columns of E are linearly independent. Thus, the matrix E is of rank d as required. By inspection, the matrix F is also of rank d.

Hence, the values of λ for which the rank of the matrix pencil $M-\lambda N$ decreases by 1 are given by

$$\lambda_k = d_{kk}^{-1} = e^{-j\phi_k}; \quad k = 1,2,\ldots,d. \tag{4.1-18}$$

The values of λ can be computed by means of a Grammian approach. The rank reducing λ's are the values of λ for which the columns of the matrix pencil $M-\lambda N$ become dependent. Define the inner product of the vectors $\underline{X}_i$ and $\underline{X}_j$ by

$$\langle \underline{X}_i, \underline{X}_j \rangle = \underline{X}_j^H \underline{X}_i \tag{4.1-19}$$

where $(\cdot)^H$ denotes the Hermitian operator. To check the dependence of the set

$$\{\underline{X}_1 - \lambda\underline{X}_2, \underline{X}_2 - \lambda\underline{X}_3, \ldots, \underline{X}_d - \lambda\underline{X}_{d+1}\}, \tag{4.1-20}$$

construct the Grammian matrix

$$G = \begin{bmatrix} \langle \underline{X}_1 - \lambda\underline{X}_2, \underline{X}_1 - \lambda\underline{X}_2 \rangle & \langle \underline{X}_1 - \lambda\underline{X}_2, \underline{X}_2 - \lambda\underline{X}_3 \rangle & \cdots & \langle \underline{X}_1 - \lambda\underline{X}_2, \underline{X}_d - \lambda\underline{X}_{d+1} \rangle \\ \langle \underline{X}_2 - \lambda\underline{X}_3, \underline{X}_1 - \lambda\underline{X}_2 \rangle & \langle \underline{X}_2 - \lambda\underline{X}_3, \underline{X}_2 - \lambda\underline{X}_3 \rangle & \cdots & \langle \underline{X}_2 - \lambda\underline{X}_3, \underline{X}_d - \lambda\underline{X}_{d+1} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \underline{X}_d - \lambda\underline{X}_{d+1}, \underline{X}_1 - \lambda\underline{X}_2 \rangle & \langle \underline{X}_d - \lambda\underline{X}_{d+1}, \underline{X}_2 - \lambda\underline{X}_3 \rangle & \cdots & \langle \underline{X}_d - \lambda\underline{X}_{d+1}, \underline{X}_d - \lambda\underline{X}_{d+1} \rangle \end{bmatrix}. \tag{4.1-21}$$

A polynomial $P(\lambda)$ whose zeros are the rank reducing numbers can be obtained from the equation

$$\det (G) = 0. \tag{4.1-22}$$

An alternative approach to the computation of the λ is given in section 2.2. It is shown that the rank reducing λ's of $M-\lambda N$ are also the generalized eigenvalues of the matrix pencil given by

$$M^H M - \lambda M^H N. \tag{4.1-23}$$

Again, once the G.E.'s of (4.1-23) are known, (2.1-2) can be used to compute the DOA's.

In summary, given N snapshots and the number of sources d, the following algorithm is proposed. It is assumed that the N snapshots are taken at a rate sufficiently fast such that $s_k(t)$, k=1,2,...,d, remain approximately constant over the N snapshots.

ALGORITHM 1

Step 1: Given N snapshots, form the averaged data

$$\hat{x}_i(t,\underline{\theta}) = \frac{1}{N} \sum_{j=1}^{N} y_i(t_j,\underline{\theta});\ i=1,2,\ldots,m. \tag{4.1-24}$$

Step 2: Form the vectors $\underline{X}_i$, i=1,2,..., d+1, in (4.1-7) by approximating

$$x_i(t,\underline{\theta}) \text{ with } \hat{x}_i(t,\underline{\theta}).$$

Step 3: Form the matrices M and N whose columns are given by $\underline{X}_i$ and $\underline{X}_{i+1}$, i = 1,2,..., d, respectively.

Step 4: Form the matrices $M^H M$ and $M^H N$.

Step 5: Compute the Generalized eigenvalues of

$$M^H M - \lambda M^H N.$$

These G.E.'s are known to be of the form

$$\lambda_k = \exp\{-j \frac{\omega_0}{c} D \sin\theta_k\}. \qquad (4.1\text{-}25)$$

Step 6: Find the DOA's using the relation

$$\lambda_k = \arcsin\{j \frac{c}{\omega_0 D} \ln \lambda_k\} \; ; \; k = 1,2,\ldots,d. \qquad (4.1\text{-}26)$$

4.1-1. Singular Signal Covariance Matrix Case:

An important property of the moving window approach is that the proposed algorithm does not fail if the source signal covariance matrix becomes singular. Singularity of the covariance matrix may be due to smart jammers using coherent signals, the presence of multipath phenomenon, etc. To verify this property consider, once again, the matrix pencil

$$M-\lambda N = E(I-\lambda\Phi)F$$

where E, Φ and F are given by (4.1-10), (4.1-11) and (4.1-14), respectively. Note that the matrix E is of rank d as long as the directions of arrival are distinct and the sensor separation is less than $\lambda/2$. The matrix F is also of rank d even in the presence of coherent sources. The diagonal matrix $(I-\lambda\Phi)$, whose $i^{\underline{th}}$ entry is $(1-\lambda e^{j\phi_i})$, is of rank d provided $\lambda \neq e^{-j\phi_i}$; i=1,2,...,d. Therefore the requirements of the pencil theorem are satisfied even when the signals are coherent. Because d rank reducing numbers exist, even in the case of coherent sources, it follows that the moving window approach is applicable even when the signal covariance matrix is singular.

4.1-2 Zero-Mean Random Signals Case:

In the moving window approach presented in section II the sources were assumed to be deterministic. Had they been random with zero mean, $x_i(t,\underline{\theta})$; i=1,2,...,m would have been zero. The matrices M and N in (4.1-8) would then be identially zero and would be of no use in determining the DOA's. In this section it is shown that a modification

of the moving window approach is applicable when the *non-zero mean* signal assumption does not hold.

Assume d random narrowband stationary sources with zero mean and center frequency ω_0. Given the vectors $\underline{Y}_i$; i = 1,2,..., d+1, where

$$\underline{Y}_1^T = \{y_1(t,\underline{\theta}),\ y_2(t,\underline{\theta}),\ldots,\ y_{m-d}(t,\underline{\theta})\}$$

$$\underline{Y}_2^T = \{y_2(t,\underline{\theta}),\ y_3(t,\underline{\theta}),\ldots,\ y_{m-d+1}(t,\underline{\theta})\}$$

$$\vdots$$

$$\underline{Y}_{d+1}^T = \{y_{d+1}(t,\underline{\theta}),\ y_{d+2}(t,\underline{\theta}),\ldots,\ y_m(t,\underline{\theta})\},$$

The inner produce $\langle \cdot,\cdot \rangle$ is now defined to be

$$\langle \underline{Y}_k,\ \underline{Y}_\ell \rangle = E[\underline{Y}_\ell^H \underline{Y}_k]. \tag{4.1-27}$$

Define the matrices

$$M_1 = \begin{bmatrix} \langle \underline{Y}_1, \underline{Y}_1 \rangle & \langle \underline{Y}_1, \underline{Y}_2 \rangle & \cdots & \langle \underline{Y}_1, \underline{Y}_d \rangle \\ \langle \underline{Y}_2, \underline{Y}_1 \rangle & \langle \underline{Y}_2, \underline{Y}_2 \rangle & & \langle \underline{Y}_2, \underline{Y}_d \rangle \\ \vdots & & & \\ \langle \underline{Y}_d, \underline{Y}_1 \rangle & \langle \underline{Y}_d, \underline{Y}_2 \rangle & \cdots & \langle \underline{Y}_d, \underline{Y}_d \rangle \end{bmatrix} \tag{4.1-28}$$

$$N_1 = \begin{bmatrix} \langle \underline{Y}_2, \underline{Y}_1 \rangle & \langle \underline{Y}_2, \underline{Y}_2 \rangle & \cdots & \langle \underline{Y}_2, \underline{Y}_d \rangle \\ \langle \underline{Y}_3, \underline{Y}_1 \rangle & \langle Y_3, \underline{Y}_2 \rangle & \cdots & \langle \underline{Y}_3, \underline{Y}_d \rangle \\ \vdots & & & \\ \langle \underline{Y}_{d+1}, \underline{Y}_1 \rangle & \langle \underline{Y}_{d+1}, \underline{Y}_2 \rangle & \cdots & \langle \underline{Y}_{d+1}, \underline{Y}_d \rangle \end{bmatrix} \tag{4.1-29}$$

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ e^{j2\phi_1} & e^{j2\phi_2} & \cdots & e^{j2\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix} \tag{4.1-30}$$

and

$$\Phi = \text{diag}\,[e^{j\phi_1}, e^{j\phi_2}, \ldots, e^{j\phi_d}].$$

Also, define the vectors

$$\underline{S}^T = [s_1, s_2, \ldots, s_d]$$

$$\underline{N}_i = [n_i, n_{i+1}, \ldots, n_{i+(m-d-1)}] \tag{4.1-31}$$

It follows that $\underline{Y}_i$ can be expressed as

$$\underline{Y}_i = A\Phi^{(i-1)}\underline{S} + \underline{N}_i. \tag{4.1-32}$$

Using (4.1-32), the inner product $\langle \underline{Y}_k, \underline{Y}_\ell \rangle$ becomes

$$\langle \underline{Y}_k, \underline{Y}_\ell \rangle = \langle A\Phi^{(k-1)}\underline{S} + \underline{N}_k, A\Phi^{(\ell-1)}\underline{S} + \underline{N}_\ell \rangle$$

$$= E[(A\Phi^{(\ell-1)}\underline{S} + \underline{N}_\ell)^H (A\Phi^{(k-1)}\underline{S} + \underline{N}_k)].$$

Assume the zero mean noise components with variance σ^2 to be uncorrelated from sensor to sensor and independent of the signals. Then

$$\langle \underline{Y}_k, \underline{Y}_\ell \rangle = \begin{cases} E[\underline{S}^H \phi^{H(\ell-1)} A^H A\phi^{(k-1)} \underline{S}] & \ell \neq k \\ E[\underline{S}^H \phi^{H(\ell-1)} A^H A\phi^{(k-1)} \underline{S}] + (m-d)\sigma^2 & \ell = k. \end{cases} \quad (4.1\text{-}33)$$

Let I_1 be defined as

$$I_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & \cdots & \cdots & 1 \\ \cdot & \cdots & \cdots & 0 \end{bmatrix}.$$

Assuming the noise power σ^2 to be known, the matrix pencil $M-\lambda N$ is defined as

$$M-\lambda N = (M_1 - (m-d)\sigma^2 I) - \lambda(N_1 - (m-d)\sigma^2 I_1).$$

Observe that the $k\ell^{\underline{th}}$ element of M is

$$E[\underline{S}^H \phi^{H(\ell-1)} A^H A\phi^{(k-1)} \underline{S}] \quad (4.1\text{-}34)$$

whereas the $k\ell^{th}$ element of N is

$$E[\underline{S}^H \phi^{H(\ell-1)} A^H A\phi^{(k)} \underline{S}]. \quad (4.1\text{-}35)$$

Let us now get a closed form expression for the expectation given in (4.1-34). To obtain this result we perform the matrix multiplications involved in (4.1-34).

$$A^H A = \begin{bmatrix} 1 & e^{-j\phi_1} & \cdots & e^{-j(m-d-1)\phi_1} \\ 1 & e^{-j\phi_2} & \cdots & e^{-j(m-d-1)\phi_2} \\ \vdots & \vdots & & \\ 1 & e^{-j\phi_d} & \cdots & e^{-j(m-d-1)\phi_d} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ e^{j\phi_1} & \cdots & e^{j\phi_d} \\ \vdots & & \\ e^{j(m-d-1)\phi_1} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix}$$

$$= \begin{bmatrix} (m-d) & 1 + \sum_{i=1}^{m-d-1} e^{j(\phi_2-\phi_1)i} & \cdots & 1 + \sum_{i=1}^{m-d-1} e^{j(\phi_d-\phi_1)i} \\ 1 + \sum_{i=1}^{m-d-1} e^{j(\phi_1-\phi_2)i} & (m-d) & \cdots & 1 + \sum_{i=1}^{m-d-1} e^{j(\phi_d-\phi_2)i} \\ \vdots & \vdots & & \vdots \\ 1 + \sum_{i=1}^{m-d-1} e^{j(\phi_1-\phi_d)i} & 1 + \sum_{i=1}^{m-d-1} e^{j(\phi_2-\phi_d)i} & & (m-d) \end{bmatrix}$$

$$\Phi^{H(\ell-1)}{}_A{}^H{}_A = \begin{bmatrix} (m-d)e^{-j(\ell-1)\phi_1} & (1+\sum_{i=1}^{m-d-1} e^{j(\phi_2-\phi_1)i})e^{-j(\ell-1)\phi_1} & \cdots & (1+\sum_{i=1}^{m-d-1} e^{j(\phi_d-\phi_1)i})e^{-j(\ell-1)\phi_1} \\ (1+\sum_{i=1}^{m-d-1} e^{j(\phi_1-\phi_2)i})e^{-j(\ell-1)\phi_2} & (m-d)e^{-j(\ell-1)\phi_2} & \cdots & (1+\sum_{i=1}^{m-d-1} e^{j(\phi_d-\phi_2)i})e^{-j(\ell-1)\phi_2} \\ \vdots & \vdots & & \vdots \\ (1+\sum_{i=1}^{m-d-1} e^{j(\phi_1-\phi_d)i})e^{-j(\ell-1)\phi_d} & (1+\sum_{i=1}^{m-d-1} e^{j(\phi_2-\phi_d)i})e^{-j(\ell-1)\phi_d} & \cdots & (m-d)e^{-j(\ell-1)\phi_d} \end{bmatrix}$$

$$
e^{j\Phi(\ell-1)}A^{H}A^{j\Phi(k-1)}x =
\begin{bmatrix}
(m-d)e^{j(k-\ell)\phi_1} & e^{-j(\ell-1)\phi_1}e^{j(k-1)\phi_2}\left(1+\sum_{i=1}^{m-d-1}e^{j(\phi_2-\phi_1)i}\right) & \cdots & e^{-j(\ell-1)\phi_1}e^{j(k-1)\phi_d}\left(1+\sum_{i=1}^{m-d-1}e^{j(\phi_d-\phi_1)i}\right) \\
e^{-j(\ell-1)\phi_2}e^{j(k-1)\phi_1}\left(1+\sum_{i=1}^{m-d-d}e^{j(\phi_1-\phi_1)i}\right) & (m-d)e^{j(k-\ell)\phi_2} & \cdots & e^{-j(\ell-1)\phi_2}\left(1+\sum_{i=1}^{m-d-1}e^{j(\phi_d-\phi_2)i}\right)e^{j(k-1)\phi_d} \\
\vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \vdots \\
e^{-j(\ell-1)\phi_d}e^{j(k-1)\phi_1}\left(1+\sum_{i=1}^{m-d-1}e^{j(\phi_1-\phi_d)i}\right) & e^{-j(\ell-1)\phi_d}e^{j(k-1)\phi_2}\left(1+\sum_{i=1}^{m-d-1}e^{j(\phi_2-\phi_d)i}\right) & \cdots & (m-d)e^{j(k-\ell)\phi_d}
\end{bmatrix}
$$

$$\phi^{H(\ell-1)}{}_A H_A \phi^{(k-1)} \underline{S} = \begin{bmatrix} s_1(m-d)\, e^{j(k-\ell)\phi_1} + s_2 e^{-j(\ell-1)\phi_1} e^{j(k-1)\phi_2}\left(1 + \sum_{i=1}^{m-d-1} e^{j(\phi_2-\phi_1)i}\right) + \\ \qquad \ldots + s_d e^{j(k-1)\phi_d}\left(1 + \sum_{i=1}^{m-d-1} e^{j(\phi_d-\phi_1)i}\right) e^{-j(\ell-1)\phi_1} \\ \\ s_1 e^{-j(\ell-1)\phi_2} e^{j(k-1)\phi_1}\left(1 + \sum_{i=1}^{m-d-1} e^{j(\phi_1-\phi_2)i}\right) + s_2(m-d)\, e^{j(k-\ell)\phi_2} + \\ \qquad \ldots + s_d e^{-j(\ell-1)\phi_2} e^{j(k-1)\phi_d}\left(1 + \sum_{i=1}^{m-d-1} e^{j(\phi_d-\phi_2)i}\right) \\ \\ \vdots \\ \\ s_1 e^{-j(\ell-1)\phi_d} e^{j(k-1)\phi_1}\left(1 + \sum_{i=1}^{m-d-1} e^{j(\phi_1-\phi_d)i}\right) + s_2 e^{-j(\ell-1)\phi_d} e^{j(k-1)\phi_2}\left(1 + \sum_{i=1}^{m-d-1} e^{j(\phi_2-\phi_d)i}\right) + \\ \qquad \ldots + s_d(m-d) e^{j(k-\ell)\phi_d} \end{bmatrix}$$

$$\underline{S}^{H}\phi^{H(\ell-1)}\ {}_{A}{}^{H}{}_{A}\ \phi^{(k-1)}\ \underline{S} =$$

$$s_1^*s_1(m-d)\ e^{j(k-\ell)\phi_1} + s_1^*s_2\ e^{-j(\ell-1)\phi_1}\ e^{j(k-1)\phi_2}\ (1 + \sum_{i=1}^{m-d-1} e^{j(\phi_2-\phi_1)i})$$

$$+ \ldots + s_1^*s_d\ e^{-j(\ell-1)\phi_1}\ e^{j(k-1)\phi_d}\ (1 + \sum_{i=1}^{m-d-1} e^{j(\phi_d-\phi_1)}) +$$

$$s_2^*s_1\ e^{-j(\ell-1)\phi_2}\ e^{j(k-1)\phi_1}\ (1 + \sum_{i=1}^{m-d-1} e^{j(\phi_1-\phi_2)i}) + s_2^*s_2(m-d)e^{j(k-\ell)\phi_2}$$

$$+ \ldots + s_2^*s_d\ e^{-j(k-1)\phi_2}\ e^{j(\ell-1)\phi_d}\ (1 + \sum_{i=1}^{m-d-1} e^{j(\phi_d-\phi_2)i}) +$$

$$s_3^*s_1 e^{-j(\ell-1)\phi_3} e^{j(k-1)\phi_1}\ (1 + \sum_{i=1}^{m-d-1} e^{j(\phi_1-\phi_3)i} + s_3^*s_2\ e^{-j(\ell-1)\phi_3}\ e^{j(k-1)\phi_2}$$

$$(1 + \sum_{i=1}^{m-d-1} e^{j(\phi_2-\phi_3)i}) + \ldots + s_3^*s_d\ e^{-j(\ell-1)\phi_3}\ e^{j(k-1)\phi_d}(1 + \sum_{i=1}^{m-d-1} e^{j(\phi_d-\phi_3)i}) -$$

$$\vdots$$

$$+ s_d^*s_1\ e^{-j(\ell-1)\phi_d}\ e^{j(k-1)\phi_1}\ (1 + \sum_{i=1}^{m-d-1} e^{j(\phi_1-\phi_d)i}) + s_d^*s_2\ e^{-j(\ell-1)\phi_d}\ e^{j(k-1)\phi_2}$$

$$(1 + \sum_{i=1}^{m-d-1} e^{j(\phi_2-\phi_d)i}) + \ldots + s_d^*s_d(m-d)\ e^{j(k-\ell)\phi_d}.$$

The above equation can be written in a much more compact form using the following sustitutions. Let

$$F_{pq} = 1 + \sum_{i=1}^{m-d-1} e^{j(\phi_p-\phi_q)i}.$$

Then

$$\underline{S}^{H}\phi^{H(\ell-1)}\ {}_{A}{}^{H}{}_{A}\ \phi^{(k-1)}\underline{S}$$

$$= \sum_{q=1}^{d}\sum_{p=1}^{d} s_q^*s_p\ e^{-j(\ell-1)\phi_q}\ e^{j(k-1)\phi_p}(1 + \sum_{i=1}^{m-d-1} e^{j(\phi_p-\phi_q)i})$$

$$= \sum_{q=1}^{d} \sum_{p=1}^{d} s_q^* s_p \, e^{-j(\ell-1)\phi_q} \, e^{j(k-1)\phi_p} \, F_{pq}.$$

Let

$$S_{pq} = E[s_q^* s_p].$$

The expectation given in (4.1-22) can finally be expressed as

$$E[\underline{S}^H \Psi^{H(\ell-1)} \, A^H \Lambda_\phi^{(k-1)} \underline{S}]$$

$$= \sum_{q=1}^{d} \sum_{p=1}^{d} S_{pq} \, e^{-j(\ell-1)\phi_q} \, e^{j(k-1)\phi_p} \, F_{pq}. \qquad (4.1\text{-}36)$$

Using this expression, M can be written as

$$M = \begin{bmatrix} \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}F_{pq} & \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}F_{pq}e^{-j\phi_q} & \cdots & \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}F_{pq}e^{-j(d-1)\phi_q} \\ \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}F_{pq}e^{j\phi_p} & \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}F_{pq}e^{-j\phi_q}e^{j\phi_p} & & \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}F_{pq}e^{j\phi_p}e^{-j(d-1)\phi_d} \\ \vdots & \vdots & & \vdots \\ \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}F_{pq}e^{j(d-1)\phi_p} & \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}F_{pq}e^{j(d-1)\phi_p}e^{-j\phi_q} & \cdots & \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}F_{pq}e^{j(d-1)\phi_p}e^{-j(d-1)\phi_q} \end{bmatrix}.$$

Using (4.1 - 23) and (4.1 - 24), matrix N becomes

$$N = \begin{bmatrix} \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq} F_{pq} e^{-j\phi_q} & \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq} F_{pq} e^{-j2\phi_q} & \cdots & \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq} F_{pq} e^{-jd\phi_q} \\ \vdots & \vdots & & \vdots \\ \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq} F_{pq} e^{-j\phi_q} e^{j(d-1)\phi_p} & \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq} F_{pq} e^{-j2\phi_q} e^{j(d-1)\phi_p} & \cdots & \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq} F_{pq} e^{-jd\phi_q} e^{j(d-1)\phi_p} \end{bmatrix}$$

To simplify further define the matrices

$$U = \begin{bmatrix} 1 & 1 & 1 \\ e^{j\phi_1} & e^{j\phi_2} & e^{j\phi_d} \\ e^{j2\phi_1} & e^{j2\phi_2} & e^{j2\phi_d} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ e^{j(d-1)\phi_1} & e^{j(d-1)\phi_2} & e^{j(d-1)\phi_d} \end{bmatrix}$$

and

$$V = \begin{bmatrix} S_{11}F_{11} & S_{12}F_{12} \cdots & S_{1d}F_{1d} \\ S_{21}F_{21} & S_{22}F_{22} \cdots & S_{2d}F_{2d} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ S_{d1}F_{d1} & S_{d2}F_{d2} \cdots & S_{dd}F_{dd} \end{bmatrix}. \qquad (4.1\text{-}37)$$

The matrix M can then be decomposed as

$$M = UVU^H . \qquad (4.1\text{-}38)$$

Similarly, the matrix N can be decomposed as

$$N = UV\Phi^H U^H. \qquad (4.1\text{-}39)$$

Thus, as required by the pencil theorem, the matrix decomposition of $M-\lambda N$ is given by

$$\begin{aligned} M-\lambda N &= UVU^H - \lambda UV\Phi^H U^H \\ &= UV\,(I - \lambda\Phi^H)U^H. \end{aligned} \qquad (4.1\text{-}40)$$

The matrices UV and U^H are of rank d as long as the directions of arrival of the signals are distinct. Hence, the values of λ for which the rank of the matrix pencil M-λN decreases by 1 are given by

$$\lambda_i = e^{j\phi_i} \; ; \; i = 1,2,\ldots, d.$$

Note that the matrices UV and U^H are of rank d even when the sources are coherent. Therefore, as in the nonzero-mean signal case, the approach does not fail in the presence of coherent sources.

The above choices for M and N are not unique. To demonstrate the flexibility of the pencil theorem, a second choice is now considered. Let

$$M_1 = \frac{1}{d} E[\underline{Y}_1\underline{Y}_j^H + \underline{Y}_2\underline{Y}_j^H + \ldots + \underline{Y}_d\underline{Y}_j^H] \tag{4.1-41}$$

and

$$N_1 = \frac{1}{d} E[\underline{Y}_2\underline{Y}_j^H + \underline{Y}_3\underline{Y}_j^H + \ldots + \underline{Y}_{d+1}\underline{Y}_j^H]; \; j=1,2,\ldots,d+1. \tag{4.1-42}$$

Let

$$\delta_{k,\ell} = \begin{cases} 1 & k = \ell \\ 0 & \text{otherwise.} \end{cases} \tag{4.1-43}$$

Using (4.1-32), M_1 and N_1 become

$$M_1 = E[A\underline{S}\underline{S}^H\Phi^{H(j-1)}A^H + A\Phi\underline{S}\,\underline{S}^H\Phi^{H(j-1)}A^H + \ldots$$

$$\ldots + A\Phi^{(d-1)}\underline{S}\,\underline{S}^H\Phi^{H(j-1)}A^H] + \sigma^2 I[1-\delta_{j,d+1}] \tag{4.1-44}$$

$$N_1 = E[A\Phi\underline{S}\,\underline{S}^H\Phi^{H(j-1)}A^H + A\Phi^{(2)}\underline{S}\,\underline{S}^H\Phi^{H(j-1)}A^H + \ldots$$

$$\ldots A\Phi^{(d)}\underline{S}\,\underline{S}^H\Phi^{H(j-1)}A^H] + \sigma^2 I[1-\delta_{j,1}]. \tag{4.1-45}$$

Let

$$M = M_1 - \sigma^2 I[1 - \delta_{j,d+1}] \tag{4.1-46}$$

and

$$N = N_1 - \sigma^2 I[1 - \delta_{j,1}]. \tag{4.1-47}$$

Define the matrices

$$S = E[\underline{S}\,\underline{S}^H], \tag{4.1-48}$$

$$T = [S + \Phi S + \ldots + \Phi^{(d-1)} S] \tag{4.1-49}$$

$$= [I + \Phi + \ldots + \Phi^{(d-1)}]S,$$

and

$$Z = \Phi^{(j-1)} A^H. \tag{4.1-50}$$

The matrix M can then be decomposed as

$$M = ATZ. \tag{4.1-51}$$

Similarly, N can be decomposed as

$$N = A \Phi T Z. \tag{4.1-52}$$

Consequently, as required by the pencil theorem, the matrix pencil M-λN can be decomposed as

$$M - \lambda N = A(I-\lambda\Phi)TZ \ . \tag{4.1-53}$$

However, this approach fails when the sources are coherent since the matrix TZ is reduced in rank for this case.

4.1-3 Prefiltering

The moving window approach yields correct results in absence of noise assuming sufficient numerical accuracy. Therefore, it is reasonable to expect that, if the SNR in the data can be improved by prefiltering, then the performance of our approach would improve. The question is, "Does prefiltering prevent application of the matrix pencil approach?" In this section we apply a frequency selective filter at each channel (see Fig. 4.2) and show that the matrix pencil decomposition is not disturbed.

Assume that a filter with impulse response h(t) is applied to each channel. The filtered output of the $i^{\underline{th}}$ channel is given by

$$u_i(t,\underline{\theta}) = \int_o^T y_i(\tau,\underline{\theta})\, h(t-\tau)d\tau \qquad (4.1\text{-}54)$$

where [o,T] is the observation interval. Using the expression for $y_i(\tau, \underline{\theta})$, $u_i(t, \underline{\theta})$ becomes

$$\begin{aligned} u_i(t,\underline{\theta}) &= \int_o^T \sum_{k=1}^{d} s_k(\tau)a_i(\theta_k)h(t-\tau)\, d\tau \\ &\quad + \int_o^T n_i(\tau)h(t-\tau)d\tau \\ &= \sum_{k=1}^{d} a_i(\theta_k) \int_o^T s_k(\tau)h(t-\tau)\, d\tau \\ &\quad + \int_o^T n_i(t)h(t-\tau)d\tau\ . \end{aligned} \qquad (4.1\text{-}55)$$

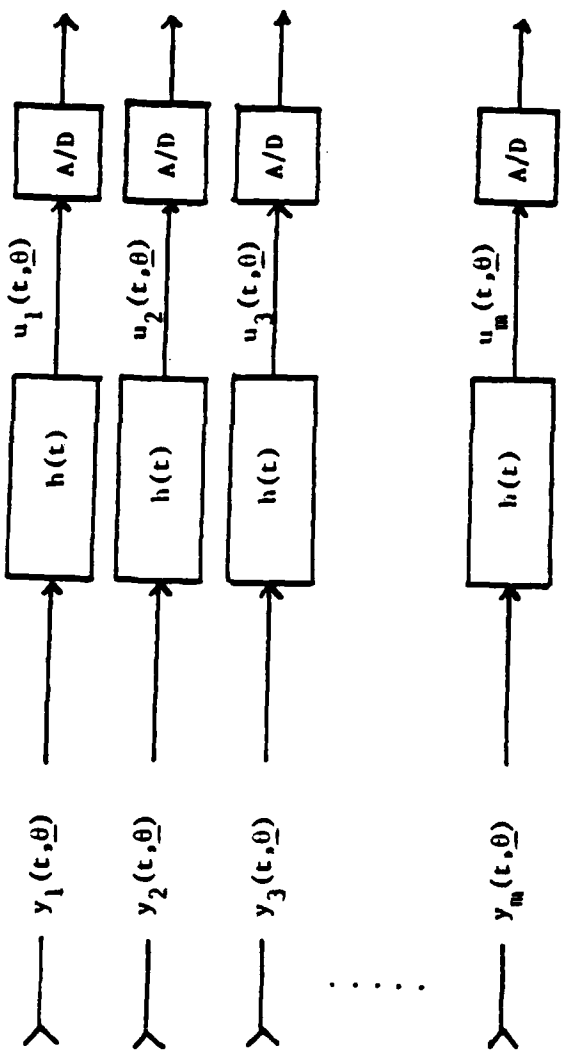

Fig. 4.2 Multichannel Prefiltering

Let

$$z_k(t) = \int_o^T s_k(\tau)h(t-\tau)d\tau \tag{4.1-56}$$

and

$$n_{u_i}(t) = \int_o^T n_i(\tau)h(t-\tau)d\tau \ . \tag{4.1-57}$$

The expression for $u_i(t,\underline{\theta})$ can then be rewritten in terms of $z_k(t)$ and $n_{u_i}(t)$ as

$$u_i(t,\underline{\theta}) = \sum_{k=1}^{d} z_k(t)a_i(\theta_k) + n_{u_i}(t). \tag{4.1-58}$$

Assuming the noise to be zero mean and the signals to be deterministic, the expected value of $u_i(t,\underline{\theta})$ is given by

$$\begin{aligned} v_i(t,\underline{\theta}) &= E[u_i(t,\underline{\theta})] \\ &= E[\sum_{k=1}^{d} z_k(t)a_i(\theta_k) + n_{u_i}(t)] \\ &= \sum_{k=1}^{d} z_k(t)a_i(\theta_k) + E[n_{u_i}(t)]. \end{aligned} \tag{4.1-59}$$

By using a rectangular window, a set of (d+1) vectors $\underline{V}_i$ is created. In particular,

$$\begin{aligned} \underline{V}_1^T &= \{v_1(t,\underline{\theta}),\ v_2(t,\underline{\theta}),\ldots,\ v_{m-d}(t,\underline{\theta})\} \\ \underline{V}_2^T &= \{v_2(t,\underline{\theta}),\ v_3(t,\underline{\theta}),\ldots,\ v_{m-d+1}(t,\underline{\theta})\} \\ &\ \ \vdots \\ \underline{V}_{d+1}^T &= \{v_{d+1}(t,\underline{\theta}),\ v_{d+2}(t,\underline{\theta}),\ldots,\ v_m(t,\underline{\theta})\}. \end{aligned} \tag{4.1.60}$$

The matrix pencil $M-\lambda N$ is formed where

$$M = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \underline{V}_1 & \underline{V}_2 & \cdots & \underline{V}_d \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \qquad N = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \underline{V}_2 & \underline{V}_3 & & \underline{V}_{d+1} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}. \tag{4.1-61}$$

Let

$$\underline{z}^T = [z_1(t), \ldots, z_d(t)]. \tag{4.1-62}$$

The vector $\underline{V}_i$ can be expressed as

$$\underline{V}_i = A_i \underline{Z} \tag{4.1-63}$$

where, as in (4.1-13), (4.1-10), (4.1-11),

$$A_i = E\Phi^{(i-1)}.$$

Define the matrix

$$F = \begin{bmatrix} z_1(t) & z_1(t)e^{j\phi_1} & \cdots & z_1(t)\,e^{j(d-1)\phi_1} \\ z_2(t) & z_2(t)e^{j\phi_2} & \cdots & z_2(t)\,e^{j(d-1)\phi_2} \\ \vdots & \vdots & & \vdots \\ z_d(t) & z_d(t)e^{j\phi_d} & \cdots & z_d(t)\,e^{j(d-1)\phi_d} \end{bmatrix}. \tag{4.1.64}$$

The matrices M and N can be expressed as

$$M = EF \quad \text{and} \quad N = E\Phi F\,. \tag{4.1-65}$$

Thus, the matrix pencil $M-\lambda N$ can be decomposed as required by the pencil theorem. Specifically,

$$M - \lambda N = EF - \lambda E\Phi F$$

$$= E\,(I - \lambda\Phi)F\ . \qquad (4.1\text{-}66)$$

The matrix decomposition is, therefore, preserved without altering the diagonal matrix Φ when multichannel filtering is added to the moving window approach. As before, the rank reducing numbers of the matrix pencil $M-\lambda N$ will still contain the necessary information to estimate the DOA's.

The purpose for using a frequency selective filter in each channel is to improve the SNR in the entries of the matrices M and N. A simple example is discussed next to illustrate the procedure. The power spectral density $P(\omega)$ of the noise is assumed to be constant up to some high cut-off frequency ω_c such that

$$P(\omega) = \begin{cases} No/2 & -\omega_c < \omega < \omega_c \\ 0 & |\omega| > \omega_c . \end{cases} \qquad (4.1\text{-}67)$$

Assume the desired signal with the largest bandwidth to have bandwidth ΔB. Let P_{s_k} denote the average power of the k^{th} desired signal. Suppose now that the received signals are put thru a rectangular bandpass filter with center frequency ω_o and bandwidth ΔB. The signal-to-noise ratio at the input of the filter is given by

$$(SNR)_i = \frac{\sum_{k=1}^{d} P_{s_k}}{N_o \omega_c} \qquad (4.1\text{-}68)$$

while the signal to noise ratio at the output of the filter is given by

$$(SNR)_o = \frac{\sum_{k=1}^{d} P_{s_k}}{N_o \Delta B} \quad . \tag{4.1-69}$$

Define the signal-to-noise ratio improvement factor to be

$$I = \frac{(SNR)_o}{(SNR)_i} \quad . \tag{4.1-70}$$

For the moving window operator and the signals posed in this example, the SNR of the entries in the matrices M and N are increased by the factor

$$I \doteq \frac{\omega_c}{\Delta B} \quad . \tag{4.1-71}$$

One can, therefore, improve the performance of the moving window approach by prefiltering the signals. The filters introduced into each channel do not disturb the matrix decomposition of $M-\lambda N$ as required by the pencil theorem. The SNR is increased in the entries of M and N and one can expect more accurate estimates of the DOA's.

4.2 Generalization to a Rectangular Planar Array

The scheme proposed in section 4.1 also lends itself to a rectangular planar array of sensors with uniform spacings d_x, d_y in the X, Y directions respectively (See Fig. 4.3).

The angular locations of d sources are again to be estimated. These estimates are to be obtained from measurements collected at the sensors of a rectangular planar array. Assume this rectangular planar array to have p sensors in the X direction and q sensors in the Y direction. Observe that one can think of this array as being comprised of either p columns of length q or q rows of length p. In this section we view the array as being composed of q rows of length p. To be able to locate d sources for this case, q has to be greater than or equal to (d+1) while p must be greater than or equal to d. Define $\underline{i}_x$ and $\underline{i}_y$ to be the unit vectors along the X and Y axes. The position of the $mn^{\underline{th}}$ sensors is given by

$$\underline{Z}_{mn} = (m-1)d_x \underline{i}_x + (n-1)d_y \underline{i}_y . \qquad (4.2\text{-}1)$$

Let $-\underline{k}_r$ be the direction of propagation for the $r^{\underline{th}}$ signal where

$$\underline{k}_r = \cos\theta_r \, \underline{i}_x + \sin\theta_r \, \underline{i}_y . \qquad (4.2\text{-}2)$$

The signal received at the $mn^{\underline{th}}$ sensor can therefore be expressed as

$$y_{mn}(t,\underline{\theta}) = \sum_{r=1}^{d} s_r(t) \exp[j \tfrac{2\pi}{\lambda} \underline{Z}_{mn} \cdot \underline{k}_r] + n_{mn}(t); \qquad (4.2\text{-}3)$$

where $\underline{Z}_{mn} \cdot \underline{k}_r$ denotes the inner product of the two vectors $\underline{Z}_{mn}$ and $\underline{k}_r$,

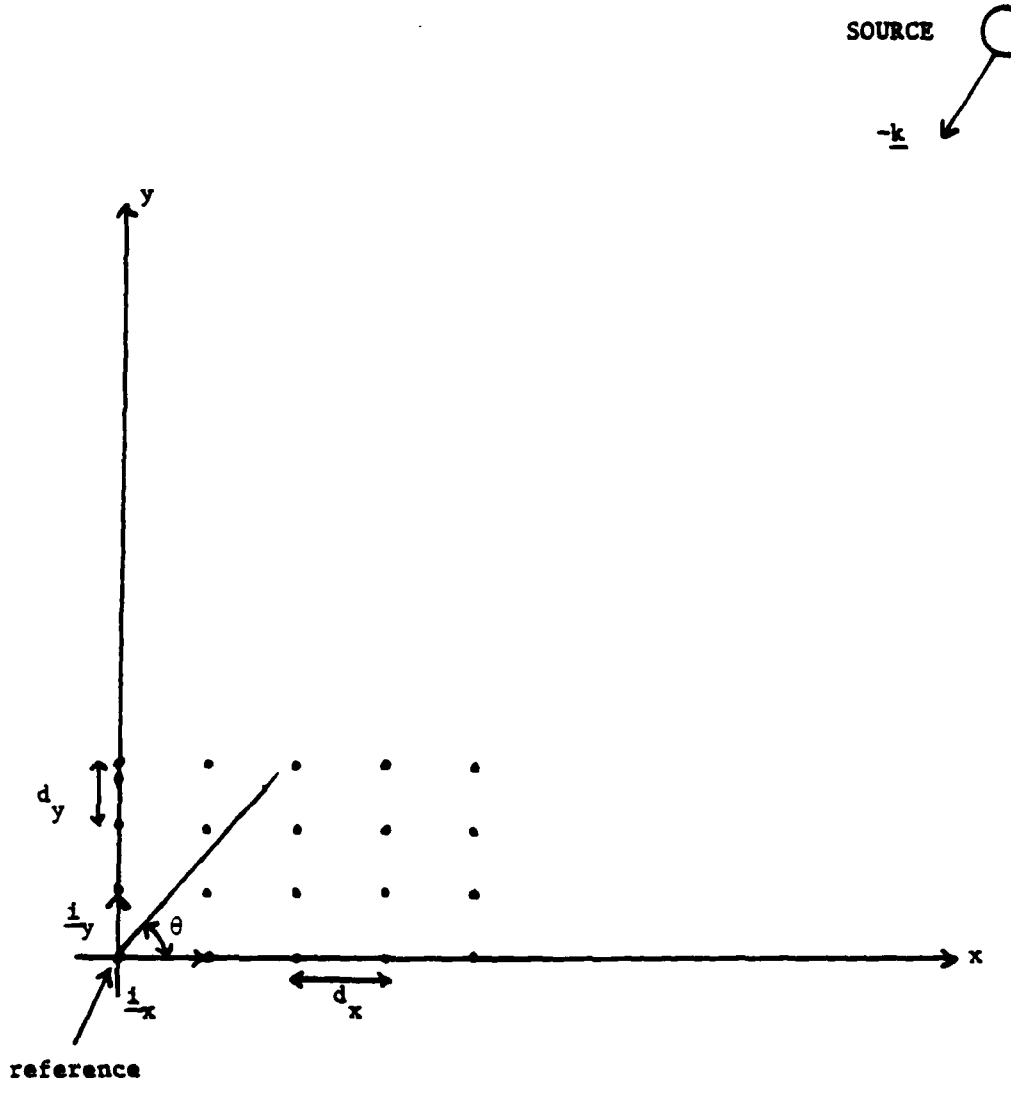

Fig. 4.3 Rectangular planar array

and $n_{mn}(t)$ denotes the additive noise at the $mn^{\underline{th}}$ sensor. Using (4.2-1) and (4.2-2), the inner product $\underline{Z}_{mn} \cdot \underline{k}_r$ results in

$$\underline{Z}_{mn} \cdot \underline{k}_r = [(m-1)d_x \underline{i}_x + (n-1)d_y \underline{i}_y] \cdot [\cos \theta_r \underline{i}_x + \sin \theta_r \underline{i}_y]$$

$$= (m-1)d_x \cos \theta_r + (n-1)d_y \sin \theta_r. \qquad (4.2\text{-}4)$$

Inserting (4.2-4) into the expression for $y_{mn}(t,\underline{\theta})$, we obtain

$$y_{mn}(t,\underline{\theta}) = \sum_{r=1}^{d} s_r(t) \exp [j\tfrac{2\pi}{\lambda} (m-1)d_x \cos \theta_r] \exp [j \tfrac{2\pi}{\lambda} (n-1)d_y \sin \theta_r] + n_{mn}(t). \qquad (4.2\text{-}5)$$

For example, the signals at the sensors in the first row of the array are given by

$$y_{11} = \sum_{r=1}^{d} s_r(t) + n_{11}(t)$$

$$y_{21} = \sum_{r=1}^{d} s_r(t) \exp [j \tfrac{2\pi}{\lambda} d_x \cos \theta_r] + n_{21}(t)$$

$$\vdots$$

$$y_{p1} = \sum_{r=1}^{d} s_r(t) \exp [j \tfrac{2\pi}{\lambda} (p-1)\, d_x \cos \theta_r] + n_{p1}(t).$$

Assuming the signals to be deterministic and the noise to be zero mean, then

$$x_{mn}(t,\underline{\theta}) = E[y_{mn}(t,\underline{\theta})]$$

$$= \sum_{r=1}^{d} s_r(t) \exp [j \tfrac{2\pi}{\lambda} d_x (m-1) \cos \theta_r] \exp[j \tfrac{2\pi}{\lambda} d_y (n-1) \sin \theta_r]. \qquad (4.2\text{-}6)$$

Using these averaged measurements, q vectors are formed as given by

$$\begin{aligned}
\underline{X}_1^T &= [x_{11}(t,\underline{\theta}),\ x_{21}(t,\underline{\theta}),\ldots,\ x_{p1}(t,\underline{\theta})] \\
\underline{X}_2^T &= [x_{12}(t,\underline{\theta}),\ x_{22}(t,\underline{\theta}),\ldots,\ x_{p2}(t,\underline{\theta})] \\
&\ \ \vdots \\
\underline{X}_q^T &= [x_{1q}(t,\underline{\theta}),\ x_{2q}(t,\underline{\theta}),\ldots,\ x_{pq}(t,\underline{\theta})].
\end{aligned} \tag{4.2-7}$$

A matrix pencil $M-\lambda N$ is then generated from these vectors where

$$M = \left[\underline{X}_1 \quad \underline{X}_2 \quad \cdots \quad \underline{X}_{q-1}\right] \qquad N = \left[\underline{X}_2 \quad \underline{X}_3 \quad \cdots \quad \underline{X}_q\right] \tag{4.2-8}$$

Let

$$\phi_{mr} = \frac{2\pi}{\lambda} d_x\,(m-1)\cos\theta_r \tag{4.2-9}$$

and

$$\psi_{nr} = \frac{2\pi}{\lambda} d_y\,(n-1)\sin\theta_r\ . \tag{4.2-10}$$

Define the matrices

$$E = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
e^{j\phi_{21}} & e^{j\phi_{22}} & & e^{j\phi_{2d}} \\
e^{j\phi_{31}} & e^{j\phi_{32}} & & e^{j\phi_{3d}} \\
\vdots & \vdots & & \vdots \\
e^{j\phi_{(p-1)1}} & e^{j\phi_{(p-1)2}} & \cdots & e^{j\phi_{(p-1)d}}
\end{bmatrix} \tag{4.2-11}$$

and

$$\Phi = \text{diag}\,[d_{11}, d_{22}, \ldots, d_{dd}]$$

$$= \text{diag}\,[e^{j\Psi_{21}}, e^{j\Psi_{22}}, \ldots, e^{j\Psi_{2d}}]. \qquad (4.2\text{-}12)$$

Also, define the vector

$$\underline{S}^T = [s_1, s_2, \ldots, s_d]. \qquad (4.2\text{-}13)$$

It can be shown that the vector $\underline{X}_i$ can be expressed as

$$\underline{X}_i = E\Phi^{(i-1)}\,\underline{S}\ ;\ i=1,2,\ldots,q. \qquad (4.2\text{-}14)$$

Define

$$F = \begin{bmatrix} s_1 & s_1 e^{j\Psi_{21}} & \cdots & s_1 e^{j\Psi_{(p-1)1}} \\ s_2 & s_2 e^{j\Psi_{22}} & \cdots & s_2 e^{j\Psi_{(p-1)2}} \\ \vdots & \vdots & & \vdots \\ s_d & s_d e^{j\Psi_{2d}} & \cdots & s_d e^{j\Psi_{(p-1)d}} \end{bmatrix}. \qquad (4.2\text{-}15)$$

The matrix M can then be expressed as

$$M = EF \qquad (4.2\text{-}16)$$

where E and F are given by (4.2-11) and (4.2-15), respectively. In a similar manner, N can be expressed as

$$N = E\Phi F\ . \qquad (4.2\text{-}17)$$

Using (4.2-16) and (4.2-17), the matrix pencil $M-\lambda N$ can be decomposed as

$$M - \lambda N = EF - \lambda E\Phi F$$

$$= E(I - \lambda\Phi)F. \qquad (4.2\text{-}18)$$

This decomposition satisfies the requirements of the pencil theorem. Since, in general, M and N are rectangular matrices, either one of the methods suggested in section 2.1 can be used to compute the generalized eigenvalues. Observe that q generalized eigenvalues exist in this case but only d of them will lie on the unit circle. These d G.E.'s are of the form

$$\lambda_i = d_{ii}^{-1} = e^{-j\Psi_{2i}} \; ; \; i=1,2,\ldots,d. \qquad (4.2\text{-}19)$$

The remaining (q-d) G.E.'s are at the origin. Thus, once the d nonzero G.E.'s are known, the DOA's can be calculated from

$$\theta_i = \arcsin \left\{ j \frac{\lambda}{2\pi d_y} \ln \Psi_{2i} \right\} \; ; \; i=1,2,\ldots,d. \qquad (4.2\text{-}20)$$

4.3 Prony's Method

In this section Prony's method [44] is presented in the context of array signal processing. We then show that Prony's algorithm is related to the moving rectangular window presented in section 4.1.

Prony's method is used for modeling data of equally spaced samples by a linear combination of exponentials. The original procedure by Baron de Prony fitted exactly d exponentials to 2d data measurements. For the case in which only an approximate fit of d exponentials to a data set of m samples is desired, where $m > 2d$, a least squares estimation procedure is used. This procedure is called the extended Prony method. The model

$$y_i(t) = \sum_{k=1}^{d} s_k(t) e^{Z_k(i-1)} + n_i(t); \quad i=1,2,\ldots,m \tag{4.3-1}$$

is to be used in the extended Prony method for approximating the measured data $y_1(t), y_2(t),\ldots,y_m(t)$. The problem is to estimate the parameters Z_k; $k=1,2,\ldots,d$. This formulation is similar to that used in array signal processing where the problem is to estimate the angular locations of d sources, $s_k(t)$ represents the complex envelopes of the signals, and Z_k are purely imaginary parameters (the damping factors are equal to zero). Z_k is given by

$$Z_k = j\phi_k = j \frac{\omega_o}{c} D \sin \theta_k \tag{4.3-2}$$

where ω_o, c, D and θ_k are defined in section 4.1. j is the square root of minus one. Because of the dependence of the model on the angles θ_k,

k=1,2,...,d, the measurements $y_i(t)$ are written as $y_i(t,\underline{\theta})$. Thus, the dual to equation (4.3-1) is

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\, e^{j\phi_{k(i-1)}} + n_i(t) \qquad (4.3\text{-}3)$$

which is identical to (4.1-3). Assuming the signals to be deterministic and the noise to be zero mean, the expected value of $y_i(t,\underline{\theta})$ yields $x_i(t,\underline{\theta})$ given by

$$x_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\, e^{j\phi_{k(i-1)}} \; ; \; i=1,2,\ldots,m. \qquad (4.3\text{-}4)$$

Finding the values of ϕ_k; k=1,2,...,d that minimize the squared error is a difficult nonlinear least squares problem. An alternative suboptimum solution was suggested by Prony.

The key to the Prony technique is to recognize that (4.3-4) is the homogeneous solution to a constant coefficient linear difference equation, the form of which is developed next. Expanding the expression for $x_i(t,\underline{\theta})$ we obtain

$$x_i(t,\underline{\theta}) = s_1(t)\, \exp[j\phi_1]\,(i-1) + s_2(t)\, \exp[j\phi_2]\,(i-1)$$
$$+\ldots+ s_d(t)\, \exp[j\phi_d](i-1). \qquad (4.3\text{-}5)$$

Using (4.3-2), (4.3-5) becomes

$$x_i(t,\underline{\theta}) = s_1(t)\, Z_1^{(i-1)} + s_2(t)\, Z_2^{(i-1)} +\ldots$$
$$\ldots+ s_d(t)\, Z_d^{(i-1)}. \qquad (4.3\text{-}6)$$

Evaluating (4.3-6) at i=1,2,...,m, we obtain

$$x_1(t,\underline{\theta}) = s_1(t) + s_2(t) +\ldots+ s_d(t)$$

$$x_2(t,\underline{\theta}) = s_1(t)Z_1 + s_2(t)Z_2+\ldots+ s_d(t)\ Z_d$$

$$\vdots \qquad (4.3\text{-}7)$$

$$x_m(t,\underline{\theta}) \quad s_1(t)Z_1^{(m-1)} + s_2(t)Z_2^{(m-1)} +\ldots+ s_d(t)\ Z_d^{(m-1)}.$$

Assume that the complex exponentials Z_k, k=1,2,...,d of (4.3-2) are the roots of the algebraic equation (also known as the prediction-error filter polynomial) given by

$$\beta_1 + \beta_2 Z + \beta_3 Z^2 +\ldots+ \beta_{d+1} Z^d = 0 \qquad (4.3\text{-}8)$$

where β_{d+1} is arbitrarily set equal to 1. In order to determine the coefficients $\beta_1, \beta_2,\ldots,\beta_d$, the first equation of (4.3-7) is multiplied by β_1, the second one is multiplied by β_2, and the $d^{\underline{th}}$ one is multiplied by β_d and finally the (d+1)th is multiplied by $\beta_{d+1} = 1$. This results in the set of equations

$$\beta_1 x_1(t,\underline{\theta}) = \beta_1 s_1(t) + \beta_1 s_2(t) +\ldots+ \beta_1 s_d(t)$$

$$\beta_2 x_2(t,\underline{\theta}) = \beta_2 s_1(t)Z_1+\beta_2 s_2(t)Z_2 +\ldots+ \beta_2 s_d(t)Z_d$$

$$\vdots$$

$$\beta_d x_d(t,\underline{\theta}) = \beta_d s_1(t)Z_1^{d-1} + \beta_d s_2(t)Z_2^{d-1} +\ldots+ \beta_d s_d(t)Z_d^{d-1}$$

$$x_{d+1}(t,\underline{\theta}) = s_1(t)Z_1^d + s_2(t)Z_2^d +\ldots+ s_d(t)Z_d^d .$$

Adding the above equations and using the fact that $Z_1, Z_2, \ldots, Z_d$ are the roots of (4.3-8), we have

$$\begin{aligned} & \beta_1 x_1(t,\underline{\theta}) + \beta_2 x_2(t,\underline{\theta}) + \ldots + x_{d+1}(t,\underline{\theta}) = \\ & s_1(t)\,[\beta_1 + \beta_2 Z_1 + \ldots + \beta_d Z_1^{d-1} + Z_1^d] \\ + \; & s_2(t)\,[\beta_1 + \beta_2 Z_2 + \ldots + \beta_d Z_2^{d-1} + Z_2^d] \\ + \; & \vdots \\ + \; & s_d(t)[\beta_1 + \beta_2 Z_d + \ldots + \beta_d Z_d^{d-1} + Z_d^d] = 0. \end{aligned} \tag{4.3-9}$$

Similarly, a set of (m-d-1) additional equations having the same form is obtained by successively starting with the second, the third,..., the (m-d)th equation. For convenience, we write $x_i(t,\underline{\theta})$ as x_i in the following matrix equation. The set of equations obtained following the procedure is written in matrix form as

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_{d+1} \\ x_2 & x_3 & \cdots & x_{d+2} \\ x_3 & x_4 & \cdots & x_{d+3} \\ \vdots & \vdots & & \vdots \\ x_{m-d} & x_{m-d+1} & \cdots & x_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4-3.10}$$

(4.3-10), which comprises (m-d) equations in d unknowns, can be solved by using a least squares approach. Having found the β_i's, the algebraic equation (4.3-8) is formed and its zeros computed. Once these zeros, Z_k; k=1,2,...,d, are known, expression (4.3-2) is used to obtain the directions of arrival θ_k, k=1,2,...,d.

The relationship between Prony's method and the moving rectangular window technique is demonstrated. These methods are related in the sense that the information needed to estimate the angular positions of the d sources is obtained by examining the dependence/independence of a set of vectors formed from the averaged vectors $\underline{X}_i$; i=1,2,...,d+1 given in (4.1-7).

Recall that the moving window approach examines the dependence of the set of vectors $\underline{X}_i - \lambda \underline{X}_{i+1}$, i=1,2,...,d given in (4.1-20). On the other hand, the algebraic equation (4.3-8), whose zeros are given by Z_k, k=1,2,...,d in (4.3-2), can be derived by examining the dependence of the vectors $\underline{X}_i$; i=1,2,...,d+1, such that

$$\beta_1 \underline{X}_1 + \beta_2 \underline{X}_2 + \ldots + \underline{X}_{d+1} = \underline{0}. \tag{4.3-11}$$

In view of (4.1-10), (4.3-11) can be written as

$$\beta_1 E\underline{S} + \beta_2 E\Phi\underline{S} + \ldots + E\Phi^{(d)} \underline{S} = \underline{0} \tag{4.3-12}$$

or, equivalently,

$$E[\beta_1 + \beta_2\Phi + \ldots + \Phi^{(d)}] \underline{S} = \underline{0}. \tag{4.3.13}$$

For equality to hold in (4.3-13), the matrix equation

$$\beta_1 + \beta_2\Phi + \ldots + \Phi^{(d)} = 0 \qquad (4.3\text{-}14)$$

must be satisfied. With reference to (4.1-11), (4.3-14) can be expanded into the following set of equations:

$$\begin{aligned}
&\beta_1 + \beta_2 e^{j\phi_1} + \ldots + e^{jd\phi_1} = 0 \\
&\beta_1 + \beta_2 e^{j\phi_2} + \ldots + e^{jd\phi_2} = 0 \\
&\vdots \\
&\beta_1 + \beta_2 e^{j\phi_d} + \ldots + e^{jd\phi_d} = 0 \ .
\end{aligned}$$

These equations are equivalent to the single polynomial equation

$$\beta_1 + \beta_2 Z + \ldots + Z^d = 0 \qquad (4.3\text{-}15)$$

where the zeros of (4.3-15) are given by

$$Z_k = e^{j\phi_k} \ ; \ k = 1,2,\ldots, d.$$

Thus, Prony's algebraic equation (4.3-8) can also be derived by studying the dependence of the set of vectors $\underline{X}_i$, i=1,2,...,d+1. As shown the DOA's can be obtained from its zeros.

We conclude that Prony's algorithm and the moving window algorithm are related in the sense that each utilizes independence of a set of vectors formed from the vectors $\underline{X}_i$; i=1,2,...,d+1.

4.4 Pisarenko's Algorithm

In this section another method, Pisarenko's algorithm [47], is presented in the context of array signal processing. Pisarenko's algorithm will also be shown to be related to the moving rectangular window.

In Prony's method one is interested in obtaining the roots of polynomial (4.3-8). Since we are interested in the roots of this polynomial, the direction of the vector $\underline{\beta}$ where

$$\underline{\beta}^T = [\beta_1, \beta_2, .., \beta_{d+1}]$$

is the important criteria and not its magnitude. In the Prony method we had set β_{d+1} to 1 which constrains the $\underline{\beta}$ vector to be on the hyperplane $\beta_{d+1} = 1$.

The goal of Pisarenko's method is to constrain the tip of the $\underline{\beta}$ vector to be on a hypersphere. Then no constraints are placed in any direction. A unit radius is chosen for the hypersphere such that

$$\sum_{i=1}^{d+1} \beta_i^2 = 1. \tag{4.4-1}$$

Without constraining β_{d+1} to 1. Pisarenko solves the set of equations (4.3-9). Given

$$E\,\underline{\beta} = \underline{0}, \tag{4.4-2}$$

where

$$E = \begin{bmatrix} x_1 & x_2 & \cdots & x_{d+1} \\ x_2 & x_3 & \cdots & x_{d+2} \\ \vdots & \vdots & & \vdots \\ x_{m-d} & x_{m-d+1} & \cdots & x_m \end{bmatrix}, \tag{4.4-3}$$

(4.4-2) is multipled by E^H to obtain

$$E^H E\underline{\beta} = \underline{0}. \tag{4.4-4}$$

Let

$$V = E^H E. \tag{4.4-5}$$

Using (4.4-5), (4.4-4) becomes

$$V\underline{\beta} = \underline{0}. \tag{4.4-6}$$

Note that V is a (d+1)x(d+1) matrix whose columns span a d-dimensional space. The smallest eigenvalue of V must therefore be equal to zero. Solving (4.4-6) for $\underline{\beta}$ is equivalent to solving for the eigenvector corresponding to the smallest (zero) eigenvalue. In particular,

$$V\underline{\beta} = \lambda_{\text{smallest}}\,\underline{\beta} = \underline{0}. \tag{4.4-7}$$

Based on this observation, Pisarenko proposed to obtain $\underline{\beta}$ by computing the eigenvector of V corresponding to the smallest eigenvalue. Once this eigenvector, which satisfies (4.4-1), is obtained, polynomial (4.3-8) is formed and its zeros are computed. These zeros, $e^{j\Phi_i}$; i=1,2,...,d, contain the necessary information to estimate the angular position of the d sources.

However, in practice, $E\underline{\beta}$ is not equal to zero but to some residual $\underline{\varepsilon}$. H. J. Price [47] solved

$$E\,\underline{\beta} = \underline{\varepsilon} \tag{4.4-8}$$

for $\underline{\beta}$ by minimizing the sums of squares of the residuals, subject to

constraint (4.4-1). He showed that the vector $\underline{\beta}$, which minimizes the squares of the residual, is the eigenvector of V corresponding to the minimum eigenvalue.

The relationship between Pisarenko's method and the moving rectangular window is demonstrated. Let $\underline{V}_j$ be the $j^{\underline{th}}$ column of V. (4.4-6) can then be rewritten as

$$\beta_1\underline{V}_1 + \beta_2\underline{V}_2 + \ldots + \beta_{d+1}\underline{V}_{d+1} = \underline{0} \ . \tag{4.4-9}$$

Since the vectors $\underline{V}_j$; j=1,2,..., d+1, are dependent, then there must exist a set of constants $\beta_i \neq 0$ such that (4.4-9) is verified. Thus Pisarenko's algorithm examines the dependence of the set

$$\{\underline{V}_1, \underline{V}_2, \ldots, \underline{V}_{d+1}\} \tag{4.4-10}$$

where

$$\underline{V}_j^T = \{<\underline{X}_j,\underline{X}_1>, <\underline{X}_j,\underline{X}_2>, \ldots, <\underline{X}_j,\underline{X}_{d+1}>\}$$

$$j = 1,2,\ldots,d+1. \tag{4.4-11}$$

Recall that $\underline{X}_j$; j = 1,2,..., d+1, are the vectors used in the moving rectangular window method. Polynomial (4.3-8) can also be obtained by examining the dependence of the set (4.4-10). Making use of (4.1-13), the $ij^{\underline{th}}$ entry of V or V_{ij} can be expressed as

$$V_{ij} = \langle \underline{X}_i, \underline{X}_j \rangle$$

$$= \langle E\Phi^{(i-1)}\underline{S}, E\Phi^{(j-1)}\underline{S} \rangle$$

$$= \underline{S}^H \Phi^{H(j-1)} E^H E \Phi^{(i-1)} \underline{S} \qquad (4.4\text{-}1$$

Using (4.4-12), $\underline{V}_i$ can be written as

$$\underline{V}_i = \begin{bmatrix} \underline{S}^H E^H E \Phi^{(i-1)} \underline{S} \\ \underline{S}^H \Phi^H E^H E \Phi^{(i-1)} \underline{S} \\ \vdots \\ \underline{S}^H \Phi^{H(d)} E^H E \Phi^{(i-1)} \underline{S} \end{bmatrix}.$$

Define the matrix R to be

$$R = \begin{bmatrix} s_1^* & s_2^* & \cdots & s_d^* \\ s_1^* e^{-j\phi_1} & s_2^* e^{-j\phi_2} & & s_d^* e^{-j\phi_d} \\ \vdots & \vdots & & \vdots \\ s_1^* e^{-jd\phi_1} & s_2^* e^{-jd\phi_2} & \cdots & s_d^* e^{-jd\phi_d} \end{bmatrix}$$

where $(\cdot)^*$ denotes the complex conjugate. The vector $\underline{V}_i$ can be expressed in terms of R as

$$\underline{V}_j = R\, E^H\, E\, \Phi^{(j-1)}\, \underline{S}. \tag{4.4-13}$$

Utilizing (4.4-13) in (4.4-9), we obtain

$$\beta_1 R E^H E \underline{S} + \beta_2 R E^H E \Phi \underline{S} + \ldots$$

$$+ \beta_d R E^H E \Phi^{(d-1)} \underline{S} + \beta_{d+1} R E^H E \Phi^{(d)} \underline{S} = \underline{0} \tag{4.4-14}$$

or

$$R E^H E [\beta_1 I + \beta_2 \Phi + \ldots + \beta_{d+1} \Phi^{(d)}] \underline{S} = \underline{0}. \tag{4.4-15}$$

For this equality to hold for an arbitrary choice of $\underline{S}$ it must be true that

$$\beta_1 I + \beta_2 \Phi + \ldots + \beta_{d+1} \Phi^{(d)} = 0. \tag{4.4-16}$$

Using (4.1-11), this matrix equation can be expanded into

$$\begin{aligned}
&\beta_1 + \beta_2 e^{j\phi_1} + \ldots + \beta_{d+1} e^{jd\phi_1} = 0 \\
&\beta_1 + \beta_2 e^{j\phi_2} + \ldots + \beta_{d+1} e^{jd\phi_2} = 0 \\
&\vdots \\
&\beta_1 + \beta_2 e^{j\phi_d} + \ldots + \beta_{d+1} d^{jd\phi_d} = 0.
\end{aligned} \tag{4.4-17}$$

This set of equation is equivalent to the single polynomial equation.

$$\beta_1 + \beta_2 Z + \ldots + \beta_{d+1} Z^d = 0 \tag{4.4-18}$$

where the zeros of (4.4-18) are given by

$$z_i = e^{j\phi_i} \quad ; \quad i=1,2,\ldots, d. \tag{4.4-19}$$

Indeed polynomial (4.4-18) which was derived by analyzing the dependence of the set (4.4-10) is the same as the polynomial given in (4.3-8).

One can then conclude that Pisarenko's algorithm and the moving window algorithm are related in the sense that each utilizes the dependence/independence of a set of vectors formed from the same vectors $\underline{X}_i$; i=1,2,...,d+1.

4.5 Comparative Performance

In this section the comparative performance of the moving window, Prony, and Pisarenko methods is evaluated by means of computer simulation.

The model used for the simulation consists of two plane waves (d=2) arising from two coherent point sources which are incident on a linear array consisting of eight (m=8) equally spaced sensor elements. The sources are assumed to be located at $\theta_1 = 18°$ and $\theta_2 = 22°$. The angular separation, $\Delta\theta = 4°$, is less than one fourth of the array's Rayleigh angular resolution which, for the given array, is about $\frac{2}{m-1} \simeq 0.28$ radians $\simeq 16.3$ degrees. Assume N snapshots are available for processing where the $j^{\underline{th}}$ snapshot is given by

$$\underline{Y}^T = \{y_1(t_j,\underline{\theta}),\ldots, y_m(t_j,\underline{\theta})\}.$$

For a given snapshot, the measurement at the $i^{\underline{th}}$ sensor is composed of signal and noise components as defined in (4.1-3). Hence,

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t) \exp\{j\frac{\omega_0}{c} D(i-1) \sin\theta_k\} + n_i(t)$$

$$= \sum_{k=1}^{d} s_k(t) \exp\{j\, 2\pi \frac{D}{\lambda} (i-1) \sin\theta_k\} + n_i(t)$$

where $n_i(t)$ is generated in the computer simulation as a zero mean, unit variance, white complex Gaussian noise. The signal portion is generated using the expression

$$\sum_{k=1}^{d} a_k(t) \exp\{j\, 2\pi \frac{D}{\lambda} (i-1) \sin \theta_k\};\ i=1,2,\ldots,m$$

where $D = \lambda/2$, $d = 2$, $\theta_1 = 18°$, $\theta_2 = 22°$ and $m = 8$. It is assumed that the N snapshots are taken at a rate sufficiently fast such that $s_k(t)$; $k=1,2$ remain approximately constant over the N snapshots. Assuming the 2 sources to be of equal power with $s_1(t) = s_2(t) = s$, the signal-to-noise ratio SNR is defined to be

$$\text{SNR} = \frac{P_s}{P_n}$$

where P_s is the signal power at sensor 1 and P_n is the noise power

$$\text{SNR} = \frac{P_s}{P_n} = \frac{4|s|^2}{2|\sigma|^2} = \frac{2|s|^2}{\sigma^2} .$$

The cases considered in the simulation are tabulated in Table 1.

SNR(dB)	$\lvert s \rvert$
30	22.36
25	12.57
20	7.07
10	2.24

TABLE 1

Using the source and receiver model as described above, the moving window, Prony, and Pisarenko methods were used to estimate θ_1 and θ_2.

Quantitative results of the three methods are given in Tables 2 and 3. 10 Monte Carlo runs of 500 snapshots each were performed for each SNR in the range of 10 - 30 dB. It can be observed from Table 2 that, at large enough SNR, the angular positions of the two sources are nicely resolved. At lower SNR, two sources are still resolved but the bias and variance of the estimates of θ_1 and θ_2 are larger. Observe that the moving window using generalized eigenvalues performs better than the Gram approach.

An unexpected result is that the Prony method and the moving window method using the generalized eigenvalue approach have identical performance. This is explained below by showing that identical equations are solved in both techniques. Let

$$\underline{X}_1 = \{x_1, x_2, \ldots, x_6\}$$

$$\underline{X}_2 = \{x_2, x_3, \ldots, x_7\}$$

$$\underline{X}_3 = \{x_3, x_4, \ldots, x_8\}.$$

Given d = 2 and the vectors $\underline{X}_1$, $\underline{X}_2$, and $\underline{X}_3$, the Prony method leads to the equation

$$\begin{bmatrix} \uparrow & \uparrow \\ \underline{X}_1 & \underline{X}_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = - \begin{bmatrix} \uparrow \\ \underline{X}_3 \\ \downarrow \end{bmatrix} \qquad (4.5\text{-}1)$$

The least squares solution of the above set of linear equations is given by

	PRONY		PISARENKO		MOVING WINDOW			
					GRAM		GEN. EIGEN.	
SNR(dB)	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$
30	18.02	22.01	18.06	21.97	17.62	22.43	18.02	22.01
25	18.01	22.06	18.13	21.93	16.94	23.17	18.01	22.06
20	17.99	22.14	18.13	22.00	15.45	24.79	17.99	22.14
10	16.06	23.82	17.24	23.07	8.22	32.48	16.06	23.82

Table 2: Mean of θ_1 and θ_2 as a function of SNR (500 snapshots, 10 Monte Carlo runs).

	PRONY		PISARENKO		MOVING WINDOW			
					GRAM		GEN. EIGEN	
SNR(dB)	σ_1^2	σ_2^2	σ_1^2	σ_2^2	σ_1^2	σ_2^2	σ_1^2	σ_2^2
30	.104	.089	.116	.092	.097	.108	.104	.089
25	.279	.308	.383	.349	.283	365	.279	.308
20	.578	1.002	.920	.852	.813	1.122	.578	1.002
10	29.05	28.37	8.54	12.14	25.09	23.60	29.05	28.37

Table 3: Variance of θ_1 and θ_2 as a function of SNR (500 snapshots, 10 Monte Carlo runs).

$$\begin{bmatrix} \leftarrow \underline{X}_1^* \rightarrow \\ \leftarrow \underline{X}_2^* \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \underline{X}_1 & \underline{X}_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = - \begin{bmatrix} \leftarrow \underline{X}_1^* \rightarrow \\ \leftarrow \underline{X}_2^* \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \underline{X}_3 \\ \downarrow \end{bmatrix} \tag{4.5-2}$$

Recall that the linear product of two vectors $\underline{X}_1$ and $\underline{X}_2$ is defined to be

$$\langle \underline{X}_1, \underline{X}_2 \rangle = \underline{X}_2^H \underline{X}_1 = X_{21}. \tag{4.5-3}$$

Using (4.5-3), (4.5-2) can be rewritten as

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = - \begin{bmatrix} X_{13} \\ X_{23} \end{bmatrix} .$$

Utilizing Cramer's rule, α_1 and α_2 are found to be

$$\alpha_1 = \frac{X_{13} X_{22} - X_{23} X_{12}}{X_{12} X_{21} - X_{22} X_{11}}$$

and

$$\alpha_2 = \frac{X_{23} X_{11} - X_{13} X_{21}}{X_{21} X_{12} - X_{22} X_{11}} .$$

This results in the quadratic equation

$$\alpha_1 \lambda^2 + \alpha_2 \lambda + 1 = 0. \tag{4.5-4}$$

Substituting for α_1 and α_2 and simplifying yields the equation

$$(X_{13}X_{22} - X_{23}X_{12})\lambda^2 + (X_{23}X_{11} - X_{13}X_{21})\lambda + (X_{21}X_{12} - X_{22}X_{11}) = 0. \quad (4.5\text{-}5)$$

The equation in the moving window approach using generalized eigenvalues is now developed. For this, the matrix pencil

$$M - \lambda N = \begin{bmatrix} \uparrow & \uparrow \\ \underline{X}_1 & \underline{X}_2 \\ \downarrow & \downarrow \end{bmatrix} - \lambda \begin{bmatrix} \uparrow & \uparrow \\ \underline{X}_2 & \underline{X}_3 \\ \downarrow & \downarrow \end{bmatrix}$$

is first formed. $M-\lambda N$ is then premultiplied by M^H yielding

$$M^H M - \lambda M^H N = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} - \lambda \begin{bmatrix} X_{12} & X_{13} \\ X_{22} & X_{23} \end{bmatrix}.$$

The generalized eigenvalues are the solution to

$$|M^H M - \lambda M^H N| = \begin{vmatrix} X_{11} - \lambda X_{12} & X_{12} - \lambda X_{13} \\ X_{21} - \lambda X_{22} & X_{22} - \lambda X_{23} \end{vmatrix}$$

$$= (X_{13}X_{22} - X_{23}X_{12})\lambda^2 + (X_{23}X_{11} - X_{13}X_{21})\lambda + (X_{21}X_{12} - X_{22}X_{11}) = 0. \quad (4.5\text{-}6)$$

Note that (4.5-5) and (4.5-6) are identical equations. Consequently, the two algorithms produce the same results.

From Table 2 it is seen that the bias and variance of all three methods is large at 10 dB. To investigate whether θ_1 and θ_2 can be better estimated using additional snapshots, 10 Monte Carlo runs were

performed with the number of snapshots increased from 500 to 5000 while SNR was maintained at 10 dB. The results are shown in Table 4. As the number of snapshots is increased, the accuracy of the estimates for both angles also increases. Correspondingly, the variance given in Table 5 decreases as the number of snapshots increase. The reason for this improvement is the well known fact that the variance of the unbiased estimator

$$\hat{x}_i(t,\underline{\theta}) = \frac{1}{N} \sum_{j=1}^{N} y_i\ (t_j,\underline{\theta})$$

decreases as 1/N thus yielding better results as N increases.

Tables 6 and 7 show the performance of these methods as $\Delta\theta$ is successively decreased to 3°, 2° and 1°. For sufficiently large SNR, the resolution capabilities of these methods compare well. Note that when $\Delta\theta$ is 1°, all methods fail to resolve the 2 sources. In this case, only one source is observed, and the estimated location of this source has a large bias and variance.

Tables 8 and 9 give the results of the three methods as the sources are kept 4° apart and are moved from broadside to endfire of the array. Toward the endfire of the array, DOA estimation becomes worse. This is because the gain of the array is greatly reduced at endfire.

Computer simulations were also carried out to investigate the performance of the moving window as given in section 4.1-2 where the signals are assumed to be random with zero mean. Its performance is compared to that of the ESPRIT algorithm under similar conditions. The

	PRONY		PISARENKO		MOVING WINDOW			
					GRAM		GEN. EIGEN.	
# of snapshots	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$
500	16.06	23.82	17.24	23.07	8.22	32.48	16.06	23.82
1500	18.47	23.63	18.02	23.08	12.31	30.37	18.47	23.63
5000	18.30	22.56	18.27	22.55	15.47	25.54	18.30	22.56

Table 4: Mean of θ_1 and θ_2 as a function of the number of snapshots (10 Monte Carlo runs, 10 dB).

	PRONY		PISARENKO		MOVING WINDOW			
					COVAR		COV. EIGEN	
# of snapshots	σ_1^2	σ_2^2	σ_1^2	σ_2^2	σ_1^2	σ_2^2	σ_1^2	σ_2^2
500	29.05	28.37	8.54	12.14	25.09	23.06	29.05	28.37
1500	3.93	17.26	2.77	3.50	3.73	27.20	3.93	17.26
5000	.64	1.57	.57	1.35	1.05	2.63	.64	1.57

Table 5: variance of θ_1 and θ_2 as a function of the number of snapshots (10 Monte Carlo runs, 10 dB).

	PRONY		PISARENKO		MOVING WINDOW			
					GRAM		GEN EIGEN	
θ_1,θ_2	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$
18,22	18.02	22.03	18.06	21.97	17.62	22.43	18.02	22.03
18,21	18.03	21.04	18.10	20.97	16.68	22.43	18.03	21.04
18,20	17.72	20.47	17.76	20.41	13.61	24.76	17.72	20.47
18,19	-6.41	27.02	13.70	24.60	-10.55	32.11	-6.41	27.02

Table 6: mean of θ_1 and θ_2 as a function of $\Delta\theta$ (500 snapshots, 10 Monte Carlo runs, 30 dB).

	PRONY		PISARENKO		MOVING WINDOW			
					GRAM		GEN. EIGEN.	
θ_1, θ_2	σ_1^2	σ_2^2	σ_1^2	σ_2^2	σ_1^2	σ_2^2	σ_1^2	σ_2^2
18, 22	.104	.089	.116	.092	.097	.108	.104	.089
18, 21	.289	.342	.388	.355	.297	.410	.289	.342
18, 20	2.203	1.583	1.937	1.424	1.627	2.243	2.203	1.583
18, 19	712.66	118.05	31.09	51.83	570.12	168.20	712.66	118.05

Table 7: variance of θ_1 and θ_2 as a function of $\Delta\theta$ (500 snapshots, 10 Monte Carlo runs, 30 dB).

	PRONY		PISARENKO		MOVING WINDOW			
					GRAM		GEN. EIGEN.	
θ_1,θ_2	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$
0,4	.02	4.05	.05	4.02	-.30	4.39	.02	4.05
8,12	7.89	11.90	7.93	11.87	7.56	12.24	7.89	11.90
18,22	18.02	22.01	18.06	21.97	17.62	22.43	18.02	22.01
28,32	27.71	31.80	27.76	31.76	27.07	32.48	27.71	31.80
36,40	35.97	39.99	36.02	39.94	34.86	41.18	35.97	39.99
40,44	39.97	43.91	40.05	43.82	38.50	45.52	39.97	43.91
46,50	45.39	49.71	45.41	49.70	43.12	52.32	45.39	49.71
54,58	53.79	56.47	53.16	57.22	48.45	63.35	53.79	56.47
62,66	61.53	65.51	61.55	65.66	-10.99	64.33	61.53	65.51
70,74	7.06	74.18	24.39	74.07	-52.16	54.61	7.06	74.18
74,78	-4.68	75.85	-8.32	75.87	-45.97	52.11	-4.68	75.85
82,86	-7.65	83.67	-18.71	83.67	-35.04	41.00	-7.65	83.67
86,90	3.68	87.17	-4.77	87.18	-35.93	40.18	3.68	87.17

Table 8: Mean of θ_1 and θ_2 as the sources move from broadside to endfire (500 snapshots, 10 Monte Carlo runs, 30 dB).

	PRONY		PISARENKO		MOVING WINDOW			
					GRAM		GEN. EIGEN.	
θ_1, θ_2	σ_1^2	σ_2^2	σ_1^2	σ_2^2	σ_1^2	σ_2^2	σ_1^2	σ_2^2
0,4	.08	.08	.08	.08	.11	.10	.08	.08
8,12	.03	.03	.04	.03	.06	.04	.03	.03
18,22	.10	.09	.11	.09	.09	.10	.10	.09
28,32	.07	.05	.07	.06	.16	.09	.07	.05
36,40	.13	.14	.16	.14	.34	.53	.13	.14
40,44	.11	.28	.13	.36	.47	1.02	.11	.28
46,50	1.03	.47	.90	.55	1.75	2.47	1.03	.47
54,58	3.01	.29	1.80	.26	5.18	9.29	3.01	.29
62,66	6.07	4.21	4.44	3.75	3852.61	237.12	6.07	4.21
70,74	3025.31	26.68	3591.77	11.28	505.57	95.99	3025.31	26.68
74,78	1277.36	.0006	2005.98	.011	821.72	504.60	1277.36	.0006
82,86	318.77	.0006	835.92	.001	1454.45	1508.38	318.77	.0006
86,90	447.56	.003	1166.39	.005	1444.15	1557.80	447.56	.003

Table 9: variance of θ_1 and θ_2 as the sources move from broadside to endfire (500 snapshots, 10 Monte Carlo runs, 30 dB).

model used for simulation is the same as the one described above except that the sources are now made incoherent. The reason for this is the ESPRIT algorithm fails for coherent sources. Making the sources incoherent gives us a fair comparison of the moving window and ESPRIT. The signals are generated by using the expression.

$$\sum_{k=1}^{2} s_k(t) \exp\{j\, 2\pi \frac{D}{\lambda} (i-1) \sin\theta_k\} \exp\{j\, \alpha_k\},$$

and the noise is generated as before. Two cases were investigated in this simulation. In case 1, $s_k(t)$ is kept constant and α_1 and α_2 are independent random phase angles uniformly distributed in the interval $[-\pi,\pi]$. In case 2, $s_k(t)$ is random with a Rayleigh distribution independent of α_1 and α_2, while α_1 and α_2 are still independent random phase angles uniformly distributed in the interval $[-\pi, \pi]$. Tables 10 and 11 show the results for case 1, and Tables 12 and 13 show those of case 2.

The matrices M and N for the moving window are formed as explained in section 4.1-2. For the ESPRIT algorithm, the first subarray is formed of the first, third, fifth and seventh sensors while the second subarray is formed of the second, fourth, sixth and eighth sensors of the linear array. These two subarrays are used to form the covariance matrices ESPRIT calls for. For every Monte Carlo run, the matrix entries for both techniques were computed from 100 snapshots. 50 Monte Carlo runs were performed using independent data sets.

One can conclude from table 10 - 13 that the moving window compares favorably to ESPRIT. The bias is slightly smaller for the moving window technique while its variance is substantially smaller.

	MOVING WINDOW		ESPRIT	
SNR(dB)	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$
30	17.99	22.02	18.04	22.04
25	18.02	22.04	18.01	22.07
20	18.18	22.07	17.32	21.91
15	18.19	22.77	17.36	22.76

Table 10: Mean of θ_1 and θ_2 as a function of SNR (100 snapshots, 50 Monte Carlo runs, constant envelope).

	MOVING WINDOW		ESPRIT	
SNR (dB)	σ_1^2	σ_2^2	σ_1^2	σ_2^2
30	.01	.01	.07	.09
25	.06	.09	.16	.24
20	.52	.55	14.02	1.25
15	4.27	4.87	16.09	20.73

Table 11: Variance of θ_1 and θ_2 as a function of SNR (100 snapshots, 50 Monte-Carlo runs, constant envelope).

	MOVING WINDOW		ESPRIT	
SNR(dB)	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\theta}_1$	$\bar{\theta}_2$
30	17.99	22.01	17.96	22.01
25	18.01	22.02	18.02	22.52
20	18.11	22.02	17.97	22.64
15	18.47	22.00	17.92	22.71

Table 12: Mean of θ_1 and θ_2 as a function of SNR (100 snapshots, 50 Monte Carlo runs, rayleigh amplitude).

	MOVING WINDOW		ESPRIT	
SNR(dB)	σ_1^2	σ_2^2	σ_1^2	σ_2^2
30	.005	.004	.03	.02
25	.02	.01	.42	12.14
20	.13	.13	.65	12.40
15	.88	1.22	1.39	12.64

Table 13: Variance of θ_1 and θ_2 as a function of SNR (100 snapshots, 50 Monte Carlo runs, rayleigh amplified).

CHAPTER 5

CONCLUSION AND SUGGESTIONS FOR FUTURE RESEARCH

5.1 Conclusion

Determination of the angular locations of d sources using an array of m sensors was the main concern of this research. The signals generated by the d sources are assumed to be narrowband.

Until recently the methods developed to deal with this problem were classified as search procedures because either

1) the algorithm solves a constrained optimization problem for each direction of look,

2) a beam is formed and its energy computed for each direction of look, or

3) a spatial correlation matrix is formed. The array manifold is then searched for the values of $\underline{a}(\theta)$ which minimize a predefined expression.

Although these search techniques may have superresolution capabilities, they are, nevertheless, computationally very complex. Because of this complexity, nonsearch procedures have been proposed. These algorithms have the following advantages over search procedures:

1) They are computationally less complex because a search procedure is not needed.

2) They do not require knowledge of element characteristics.

3) They do not require a calibration of the array. This completely eliminates need for storage of the array manifold which

can be very large for multidimensional problems.

4) They may not require knowledge of the array geometry, as is the case with ESPRIT.

A generalized formulation is proposed for these nonsearch procedures. This formulation consists of forming a matrix pencil $M-\lambda N$ and computing its rank reducing numbers. The rank reducing numbers are the generalized eigenvalues for the case of a square matrix pencil. For the case of a rectangular matrix pencil, the rank reducing numbers can be obtained by either using the Grammian approach or by transforming the problem into a generalized eigenvalue problem. The pencil theorem given in chapter 2 establishes the relationship between the rank reducing values of λ and a functional form $f(\phi_i)$ which is a nonlinear function of the angular position of the $i^{\underline{th}}$ source; i=1,2,...,d. The form of $f(\phi_i)$ is determined by the operators applied to the measurements.

Three different operators are presented in this dissertation: the phase delay operator, the summation operator, and the moving window operator. All three methods are analyzed and formulated in terms of the generalized formulation. The matrix decomposition required by the pencil theorem heps to explain why ESPRIT fails for coherent signals. In particular, the rank requirement fails when the signals are coherent. The moving window and the summation operator were shown not to violate any of the pencil theorem requirements and thus do not fail for coherent signals.

The flexibility of the matrix pencil approach allowed us to formulate the moving window operator for the case of deterministic signals as

well as for the zero mean random case. Two well known methods in the field of system identification, namely Prony's and Pisarenko's methods, are applied to the problem of direction finding. Their relationship to the moving window technique is demonstrated. All three techniques extract the information about the angular locations of the d signals by examining the dependence/independence of a set of vectors. Each technique derives the set of vectors as a transformation of the same (d + 1) vectors.

The nonsearch methods presented in this research are capable of making high resolution DOA estimation. These methods have great potential because of their computational simplicity. They are easy and cheap to implement.

As one develops an idea, there are always new questions and problems that arise in the process. In the following section suggestions are made for future research.

5.2 Suggestions for Future Research

The generalized approach is, as its name indicates, a very general concept. The rank reducing numbers, $\lambda_1, \lambda_2, \ldots, \lambda_d$, of the matrix pencil $M-\lambda N$ are related to the directions of arrival thru a functional form that depends on the operator applied to the measurements. The possibility of using operators other than those discussed in this dissertation remains to be explored.

The formulation of different methods in terms of a common framework has the advantage that their performance can be compared within

the generalized approach. For example, suppose two methods are formulated in terms of the matrix pencil. Processing of the matrix pencil is identical for both methods. However, the entries in the pencil will differ. A statistical analysis of these entries can then be carried out. The method whose entries have smaller bias and smaller variance will perform better.

We know from the design of FIR filters that different windows have different characteristics. The window used in chapter 4 is a rectangular window. It can very easily be shown that any shape window would work. The question is, "How do differently shaped windows affect the performance of the moving window operator?"

It was shown in section 4.1-3 that prefiltering can be included into these nonsearch procedures without disturbing the matrix decomposition required by the pencil theorem. An important issue concerning prefiltering is the design of a suitable filter. If one knew enough about the noise and the desired signal, one would be able to build a filter to improve the signal-to-noise ratio and, therefore, improve the performance of the technique.

The signals in direction finding, spectral estimation, system identification, and adaptive arrays can all be modeled as a sum of exponentials. Consequently, effort should be devoted to applying the results of this disertation to those other areas.

Another issue that should be given consideration is sensor coupling. The effect of sensor coupling has not been discussed in the open literature on direction finding. Investigation into this problem might help improve existing algorithms.

The generalized framework developed in this research relies on the computation of the rank reducing numbers of a matrix pencil $M-\lambda N$. An efficient algorithm [48] based on the Schur decomposition has been developed for the case of a square matrix pencil. However, for the case of a rectangular pencil, no efficient algorithm has been shown for explicitly computing the coefficients of the polynomial $P(\lambda) = \det(G) = 0$. Preliminary work suggests that such an approach is possible [42, 47].

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